

THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY

R. L. GOODSTEIN

WITH THE ASSISTANCE OF

H. M. CUNDY K. M. SOWDEN

DECEMBER 1956

Vol. XL No. 334

A Class of Configurations and the Commutativity of Multiplication. M. W. Al-Dahir	page 241
What are x and y ? Karl Menger	246
Some Fundamentals of Space Mechanisms. N. Rosenauer	256
The Inverting Top. D. G. Parkyn	260
Correspondence. A. W. Siddons ; C. V. Durell	266
Mathematical Notes (2631-2660). W. N. Bailey ; J. C. Barton ; A. E. Berriman ; A. N. Black ; F. van der Blij ; H. M. Cundy ; J. Ginsburg ; F. M. Goldner ; R. Goormatigh ; A. Hurrell ; J. Leech ; E. H. Lloyd ; J. Massalski ; G. Matthews ; O. D. Mitrinović ; E. H. Neville ; A. R. Pargeter ; B. M. Peck ; D. Perčinková-v'Čkova ; H. Perfect ; P. Stein ; G. N. Watson ; A. Wilanski ; L. C. Woods.	267
Reviews. T. A. A. Broadbent ; B. C. Brookes ; H. M. Cundy ; C. V. Durell ; R. L. Goodstein ; C. A. Haywood ; S. Inman ; A. M. Macbeath ; K. Mahler ; D. Pedoe ; A. Prag ; E. J. F. Primrose ; F. Sandon ; F. R. Shaw ; B. A. Swinden ; F. J. Tongue ; T. J. Willmore ; R. C. H. Young	301
Gleanings Far and Near (1865-1871)	245
Address of the Mathematical Association and of the Hon. Treasurer and Secretaries	320

5s. 6d. net

G. BELL AND SONS LTD
PORTUGAL STREET · LONDON · W.C.2



G. BELL & SONS, LTD., PORTUGAL STREET, LONDON, W.C.2

SCHOOL MATHEMATICS

by H. E. PARR, M.A. Part I. 8th Edition. 8s. 9d.; with answers, 9s. 3d. Or in two sections, A and B, 5s. each. Part II. 7th Edition. 11s.; with answers, 11s. 6d. Or in two sections, A and B, 6s. 6d. each. Part III. 5th Edition. 6s. 6d.; with answers, 7s.

"A book which many teachers will want to try with their classes . . . the bookwork and examples are models of their kind. The type and layout are excellent, the many illustrative diagrams are bold and clear."—JOURNAL OF EDUCATION.

A FIRST GEOMETRY

by C. V. DURELL, M.A. 6th Edition. 224 pages. 6s.

This book provides a two-year course of Stage A geometry for the years 11+ to 13+. "Ideas are introduced gradually. . . . The exercises are very numerous and afford profuse matter for detailed discussions. . . . Throughout, the eye and mind are assisted by numerous diagrams. . . . There are many stimulating questions."—MATHEMATICAL GAZETTE.

MECHANICS

by P. GANT, B.Sc. 3rd Edition. Parts I-III, 9s. Part IV, 5s. 6d.

"I regard the layout of the bookwork and worked examples as very good and the 'discussions' at the end of the worked examples are admirable."—A SENIOR MATHEMATICS MASTER.
"Clear and direct with a welcome absence of emphasis on formulae."—HIGHER EDUCATION JOURNAL.

MATHEMATICAL TABLES

compiled by C. V. DURELL, M.A. 22nd Edition. 1s. 6d.

"The 'Difference Columns' in many books of tables give a fallacious appearance of accuracy at times when the differences are changing rapidly. Accordingly, in this book, where necessary, the tables give the average difference for $1'$, calculated over 12' intervals instead of intervals of 1° . The use of these difference columns is just as easily learnt and their superior accuracy is evident."—JOURNAL OF EDUCATION.

THE
MATHEMATICAL GAZETTE

EDITED BY

PROF. R. L. GOODSTEIN, UNIVERSITY COLLEGE OF LEICESTER

WITH THE ASSISTANCE OF

DR. H. MARTYN CUNDY, THE BEECHES, OBORNE ROAD, SHERBORNE, DORSET

MISS K. M. SOWDEN, NEWTON PARK COLLEGE, NEWTON ST. LOE, BATH

VOL. XL

DECEMBER, 1956

No. 334

A CLASS OF CONFIGURATIONS AND
THE COMMUTATIVITY OF MULTIPLICATION

BY M. W. AL-DHAHIR

1. *Definitions.* A configuration is a finite collection of points, lines and planes with a number of each on each ; any one of the three kinds may be empty.

A configuration A^* is called a "natural extension" of a configuration A , if the elements of A^* consist of those of A plus those elements which may be obtained by successive operations of joining and intersecting among the elements of A and their derivatives ; A is called a "natural contraction" of A^* . If a configuration A is obtained from a configuration B with the help of a certain number of natural extensions and contractions, then A and B are called "naturally equivalent". Clearly, natural equivalence, between two configurations, is an equivalence relation.

2. *Möbius Tetrads.* In 1828, F. Möbius [3] proved the following theorem : Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two tetrahedra such that the vertices B_1 , B_2 , B_3 , B_4 are incident to the planes $(A_2A_3A_4)$, $(A_1A_3A_4)$, $(A_1A_2A_4)$ and $(A_1A_2A_3)$ respectively. If, in addition, the vertices A_1 , A_2 , A_3 are incident to their corresponding faces of the second tetrahedron, then A_4 must be incident with the plane $(B_1B_2B_3)$ provided the theorem of Pappus holds. Schönhardt [5], answering a question of Reidemeister [4], proved that the converse is also true.

Let us describe two tetrahedra satisfying the hypothesis of the above theorem of Möbius as being in an incomplete M -relation. Then the theorem may be directly related to the commutative identity of multiplication as follows : Let S_2 be a projective 3-space defined (analytically) over a division ring R . Every incomplete M -configuration is in fact complete if, and only if, R is commutative.

(1) Let R be commutative. We may write :

$$\begin{aligned} B_1 &= K_{12}A_2 + K_{13}A_3 - A_4 \\ B_2 &= K_{21}A_1 + K_{23}A_3 - A_4 \\ B_3 &= K_{31}A_1 + K_{32}A_2 - A_4 \\ B_4 &= \quad A_1 + \quad A_2 + A_3. \end{aligned}$$

Since A_1 is on $(B_2B_3B_4)$, we may write $\lambda_1B_2 + \lambda_2B_3 + \lambda_3B_4 = A_1$, giving $K_{23} = -K_{32}$. Similarly, we get $K_{12} = -K_{21}$, $K_{13} = -K_{31}$. Now it is a simple matter to check that the determinant $\begin{vmatrix} B_1 & B_2 & B_3 & A_4 \end{vmatrix}$ vanishes.

(2) Let a and b be any two elements of R different from 0 and 1 and so that $a+b \neq 1$.

Let $T_1 : A_1A_2A_3A_4$ be the tetrahedron of reference. We find the points :

$$\begin{aligned} B_1 &= aA_1 - bA_2 + A_4 \\ B_2 &= -aA_1 + A_2 + A_4 \\ B_3 &= +bA_1 - A_2 + A_4 \\ B_4 &= +A_1 + A_2 + A_3. \end{aligned}$$

By left-multiplication only, one may verify that the two tetrahedra, $T_1 : A_1A_2A_3A_4$ and $T_2 : B_1B_2B_3B_4$ are in an incomplete M -relation. By the Möbius proposition, the relation is complete, and hence $A_4 = \mu_1B_1 + \mu_2B_2 + \mu_3B_3$. This implies $\mu_2a = \mu_3b$, $\mu_1a = \mu_3$ and $\mu_1b = \mu_2$. Combining these equations, we get $ab = ba$. Hence R is commutative.

The question arises whether it is possible to prove a special, but non-degenerate, form of the Möbius theorem without the use of the commutativity of multiplication. This question will be answered in the following two theorems.

Theorem 1.

Let S_3 be a projective 3-space defined over a division ring. Let $T : 1234$ and $T' : 1'2'3'4'$ be two tetrahedra in an incomplete M -relation where the incidence $(1'2'3'4')$ is hypothetically lacking. Let two adjacent pairs of the corresponding edges of the two tetrahedra, say $\{12, 1'2'\}$ and $\{13, 1'3'\}$, be dependent. Then the M -configuration is, in fact, complete.

Proof.

Let 1234 be the reference tetrahedron. According to the previous discussion, the vertices of T' may be denoted as follows :

$$\begin{array}{cccc} 1' & 2' & 3' & 4' \\ (0, a, -b, 1), & (-a, 0, c, 1), & (b, -c, 0, 1), & (1, 1, 1, 0). \end{array}$$

Using the condition that (12) and $(1'2')$ are coplanar, we get $c = -b$. From the fact that (13) and $(1'3')$ are coplanar, we derive that $c = -a$. Hence, $b = a = -c$. Now it is an easy matter to show, without using commutativity, the dependence of the set $(1'2'3'4')$.

Theorem 2.

In a projective 3-space defined over a division ring, let T and T' be two tetrahedra in an incomplete M -relation. If two pairs of the correspondingly adjacent edges are coplanar, so are the oppositely remaining such pairs.

Proof.

Let 1234 be the reference tetrahedron. By the previous theorem, the vertices of T' may be denoted as follows :

$$\begin{array}{cccc} 1' & 2' & 3' & 4' \\ (0, a, -a, 1), & (-a, 0, -a, 1), & (a, a, 0, 1), & (1, 1, 1, 0). \end{array}$$

Hence, by direct and simple computation one proves the dependence of the pairs $\{34, 3'4'\}$ and $\{24, 2'4'\}$. In fact, considerably more than this is true. It is easy to verify that the following sets of four lines are concurrent : $12, 1'2', 34', 3'4'$; $13, 1'3', 24', 2'4'$; $24, 2'4', 13', 1'3'$; $34, 3'4', 12', 1'2'$. A diagram of this configuration in the case where the coordinates are barycentric and $a = 1/3$ is shown in fig. 1.

The Eight-lines Theorem.

3. H. F. Baker [1] has shown that the theorem of Pappus in the plane is equivalent to the following "Eight Lines" theorem in 3-space, which it is convenient to formulate in the following self-dual form: If fifteen of the sixteen pairs of lines that can be formed from two quadruples of mutually skew-lines are non-skew, so is the sixteenth pair.

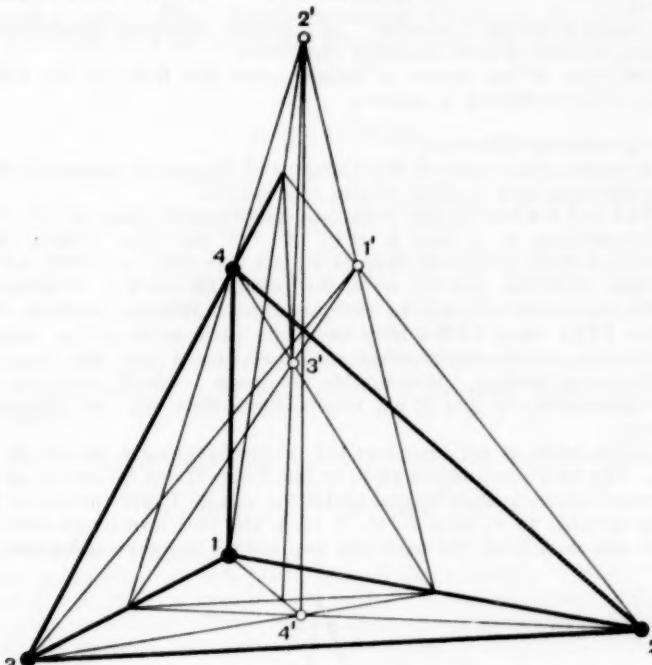


FIG. 1.

We remark that the term "non-skew" is self-dual. It means that the two lines are incident to a point and a plane. In what follows we relate the above proposition to the theorem of Möbius directly [compare 2].

Let I, II, III, IV and I' , II' , III' , IV' be two quadruples of mutually skew-lines. Let $I \cdot I' = 1$, $II \cdot II' = 2$, $III \cdot III' = 4$, $IV \cdot IV' = 3$, $I \cdot III' = 2'$, $II \cdot III' = 1'$, $III \cdot IV' = 3'$. We shall prove, assuming the axioms of incidence, that IV intersects IV' if, and only if, the theorem of Möbius holds.

(1) Suppose IV' intersects the plane (123) in $4'$. The two tetrahedra 1234 and $1'2'3'4'$ form an incomplete Möbius configuration. Hence, the point $4'$ is on the plane $(1'2'3)$, and therefore $4'$ is on the line of intersection of the planes (123) and $(1'2'3)$ which is IV .

(2) Let 1234 and $1'2'3'4'$ be two tetrahedra in an incomplete M -relation where the incidence $(1234')$ is hypothetically lacking. The following two quadruples of mutually skew-lines satisfy the hypothesis of the "Eight-lines" theorem:

$$\{14', 1'4, 2'3, 23'\}; \{14, 1'4', 23, 2'3'\}.$$

Therefore, the line (23) must intersect the line $(14')$, which means that the point $4'$ lies on the plane (123) .

By analysis of the preceding proof, one obtains the following two theorems.

Theorem 3.

The configuration of Möbius is "naturally equivalent" to the "Eight-lines" configuration.

Theorem 4.

There exists a certain "number" of mutually inscribed tetrahedra whose vertices lie on a set of four mutually skew-lines.

We note that if the space is defined over the field of the reals, the "number", in theorem 4, is infinite.

The Two-Quadrangle Theorem.

4. It is known [6, 1] that if the theorem of Pappus is assumed, then the following theorem may be derived and conversely:

Let 1234 and $1'2'3'4'$ be two complete quadrangles, lying in two different planes intersecting in a line L (Fig. 2). If the five points, namely, $(12) \cdot (3'4') = A$, $(23) \cdot (1'4') = B$, $(34) \cdot (1'2') = C$, $(14) \cdot (2'3') = D$, $(24) \cdot (1'3') = E$, are incident with the line L , so is the point $(13) \cdot (2'4')$. It is natural to relate this statement directly to the theorem of Möbius. Indeed, the two tetrahedra $1234'$ and $1'2'3'4$ may be easily seen to be in an incomplete Möbius-relation, where the incidence of the point 1 and the plane $(2'3'4)$ is hypothetically lacking. Hence, this incidence holds if, and only if, the line (13) intersects the line $(2'4')$, which shows that the two theorems are equivalent.

This result leads to an easy method of constructing a model of Möbius Tetrads. The two quadrangles and the line l (fig. 2) can be drawn on a sheet of pasteboard which is then hinged about the line l . If the vertices $1, 2, 3$ are joined by threads to $4'$, and $1', 2', 3'$ to 4 , the two tetrahedra will appear. The card can be folded flat with the two halves together when not in use.

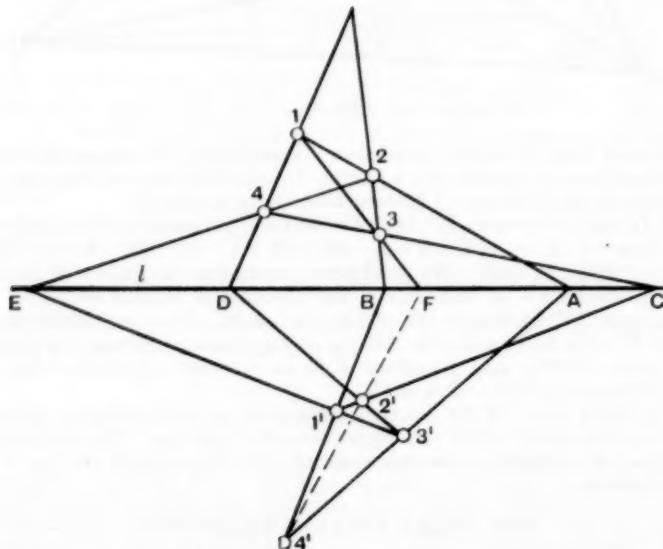


FIG. 2.

The threads should all be taut when the two halves are approximately at right angles.

Summarizing this result, we obtain

Theorem 5.

The *M*-configuration is "naturally equivalent" to the two-quadrangle configuration.

Acknowledgement. The author's thanks are due to Dr. Martyn Cundy for drawing the diagrams and preparing the manuscript for the press.
Baghdad

M. W. AL-D.

REFERENCES

1. H. F. Baker, *Principles of Geometry*, Vol. I, Cambridge (1925).
2. H. S. M. Coxeter, "Self-Dual Configurations and Regular Graphs", *Bull. Amer. Math. Soc.*, Vol. 56 (1950), pp. 413-455.
3. A. F. Möbius, *Kann Von Zwei dreiseitigen pyramiden eine jede in Bezug auf die andere um-und eingeschrieben zugleich heissen?* *Crelle* 3 (1828).
4. K. Reidemeister, "Zur Axiomatik der 3-dimensionalen projektive Geometrie", *Jhbr. Deutsch. Math. Verein*, 38 (1929), p. 71.
5. Schönhardt, *Jhbr. Deutsch. Math. Verein*, 40 (1931), pp. 48-50.
6. O. Veblen and J. W. Young, *Projective Geometry*, Boston, New York, Vol. I (1910).

GLEANINGS FAR AND NEAR

1865. THE EPIMENIDES.

It was enquired what was the origin of the sophistry "When I lie and admit that I lie, do I lie or speak the truth?"—Aulus Gellius: *Attic Nights*, XVIII, II Loeb edition, III, p. 301. (Many other references in Weyl, *Philosophy of Mathematics and Natural Science* 1949, p. 228.) [Per Prof. H. G. Forder.]

1866. "Seeing that he (Sir William Thomson, Lord Kelvin) once remarked to me about an answer I was suddenly called to give, 'Now, really, I don't think I have ever heard anything so foolish,' it may be supposed that I was a poor pupil. Indeed the whole realm of physics and mathematics was to me, then and always, unintelligible."

He found it indeed not only unintelligible, but almost intolerably tedious; and one day, at home, when he was grappling with some mathematical problem, he yawned so widely that his jaws were locked. They remained immovable and Cosmo speechless until a doctor came in and put him right.—J. G. Lockhart, *Cosmo Gordon Lang*, p. 11. [Per Rev. A. F. MacKenzie.]

1867. "I left accepted for matriculation and hopeful of a scholarship in the following October. But when I discovered that if, in those days, I intended to read for a Tripos, I would have to pass what I think were called 'Additional Subjects', involving much mathematics, I turned away, faithless to my first enthusiasm, from the prospect of wading to the Moral Sciences or History Tripos through weary bogs of my hated mathematics."

When in 1920 he went to Cambridge to receive the honorary degree of LL.D., he was able to twit his hosts that he had got it without any mathematics.—J. G. Lockhart, *Cosmo Gordon Lang*, p. 23. [Per Rev. A. F. MacKenzie.]

WHAT ARE x AND y ?

BY KARL MENGER

Illinois Institute of Technology

Introduction [1]

1. Consistent Classes of Quantities. An ordered pair whose second member (or *value*) is a number, while its first member (or *object*) may be anything, will be referred to as a *quantity*. By a *consistent class of quantities*—briefly, c.c.q.—we mean a class of quantities that does not contain two quantities with equal objects and unequal values. Reviving Newton's term, we will refer to c.c.q.'s also as *fluents*. The class of all objects (of all values) of the quantities belonging to a fluent is called the *domain* (the *range*) of the fluent.

Given a plane and a unit of length, any pair $(\kappa, r(\kappa))$ consisting of a circle κ and its radius $r(\kappa)$ is a quantity. The class of all these pairs for any circle κ is a c.c.q.—the radius r . The following three classes, for various reasons, are not c.c.q.'s: that of all pairs $((r(\kappa), \kappa)$ for any circle κ ; that of all pairs $(\kappa, \zeta(\kappa))$ for any κ , where $\zeta(\kappa)$ is the center of the circle κ ; that of all pairs of numbers.

If, for a positive integer n , the objects of all quantities belonging to a fluent are n -tuples of numbers [2] then the fluent is called an *n-place function*. An example of a 2-place function is the class of all pairs $((a, b), a)$ for any two numbers a and b ; another is the class of all pairs $((a, b), b)$. For obvious reasons, these functions might be called the *2-place selectors*; the first will be denoted by I , the second by J ; their values for (a, b) by $I(a, b)$ (which equals a) and $J(a, b)$ (which equals b). Examples of 1-place functions are the class of all pairs (c, c) for any number c , and the class of all pairs $(x, \log x)$ for any $x > 0$. They are called the *identity* and the *logarithmic function*. The former will herein be denoted by j , its value for c , by $j(c)$, which equals c .

A class of pairs of functions that does not contain two pairs with equal first and unequal second members is called an *operator*. An example is the class of all pairs (f, Df) for any everywhere differentiable function f , where Df denotes the derivative of f .

2. Variables. By a *variable* we mean a symbol that, in some context and according to definite stipulations, may be replaced by the designation of any element of a certain class (called the *scope* of the variable). In the preceding definitions, the letters c and x are *numerical variables*, i.e., symbols that in the pairs mentioned may be replaced by numerals such as 3, and then yield pairs such as $(3, 3)$ and $(3, \log 3)$ belonging to the identity and the logarithmic functions, respectively. In those definitions, the scope of c is the class of all numbers, that of x is the class of all positive numbers. The letter f in the definition of the operator is a *function variable*, i.e., a variable whose scope is a class of functions.

It is common to all variables that they may be replaced by any other symbol of the same kind without any change of the meaning. For instance, the logarithmic function may be defined as the class of all pairs $(c, \log c)$ for any $c > 0$; and the operator mentioned, as the class of all pairs (g, Dg) for any everywhere differentiable function g .

3. Remarks. Some mathematicians (if hardly any scientists) call all fluents "functions". But only fluents whose domains consist of numbers or systems of numbers have the power to connect c.c.q.'s. The logarithm connects the radius r with $\log r$, which is the class of all pairs $(\kappa, \log r(\kappa))$ for any circle κ ; similarly, the area a with $\log a$; the exponential function with the identity function; the function \cos with $\log \cos$; etc. The radius r lacks the

power to connect two fluents. There is no radius of the area nor a radius of the cosine. Nor can fluents other than those of the type of \log be differentiated or integrated : $dr/d\kappa$ and $\int r \, d\kappa$ are meaningless. Hence a special name for those fluents whose domains consist of numbers or systems of numbers is practically indispensable, and no name for them is more appropriate than "functions".

Many scientists and some mathematicians call fluents "variable quantities" — often simply "variables". The latter usage is quite unfortunate. It certainly has not been conducive to the maintenance of a clear and sharp distinction between what above have been called fluents (specific classes !) and variables (replaceable symbols!). Yet this distinction is of the utmost importance in pure and applied mathematics if terms and symbols are to be used according to articulate rules.

A Question with a Dozen Answers.

Two symbols are ubiquitous in mathematical writings—the letters [3] x and y . What is their meaning? The answer altogether depends upon the context.

I. *Numerical Variables*. In the following statements, definitions, and problems, x and y are numerical variables.

(1) The class of all pairs $(x, x+1)$ for any x contains an element not belonging to the

the class of all pairs $\left(x, \frac{x^2 - 1}{x - 1}\right)$ for any $x \neq 1$,

namely, the pair $(1, 2)$; and one element, namely, $(2, 3)$, not belonging to

the class of all pairs $\left(x, \frac{x^2 - x - 2}{x - 2}\right)$ for any $x \neq 2$.

(2) In the realm of integers mod. 2, the class of all pairs (x, x) for any x , and the class of all pairs (x, x^2) for any x , are equal.

(3) Let j^{-1} denote the class of all pairs (x, x^{-1}) for any $x \neq 0$.

(4) Let L denote the class of all pairs (x, y) such that $2x + 3y = 5$. Obviously equivalent is the definition of L as the class of all pairs (y, x) such that $2y + 3x = 5$. The class L' of all pairs (x, y) such that $2y + 3x = 5$ is different from L .

(5) $x^2 - 9y^2 = (x + 3y) \cdot (x - 3y)$ for any x and y ;

$y^2 - 9x^2 = (y + 3x) \cdot (y - 3x)$ for any x and y (or any y and x).

(The two (or three) preceding statements are identical in meaning.)

(6) $(x + yi)^2 = x^2 - y^2 + 2xyi$ for any x and y .

(7) $\sin 2x = 2 \sin x \cos x$ for any x .

(8) $D \sin x = \cos x$ for any x .

(9) $D \log x = j^{-1}(x)$ for any $x > 0$.

(10) If $F(x, y) = x^3y^5$ for any x and y ,

then $D_1F(x, y) = 3x^2y^5$ and $D_2F(x, y) = 5x^3y^4$ for any x and y .

(11) Find all numbers x such that $x^2 - 1 = 0$.

(12) Find all pairs of numbers x, y such that $2x + 3y = 5$ and $x + y = 3$.

As these examples show, numerical variables are put to a variety of uses. Theorems I(5)–(10) stipulate that a replacement of the variables by numerals yields a formula expressing a valid connection of specific numbers; e.g., I(5) yields $5^2 - 9 \cdot 7^2 = (5 + 3 \cdot 7) \cdot (5 - 3 \cdot 7)$. In each definition involved in I(1)–(3), such a replacement results in an element of the class defined; e.g., $(3, 3^{-1})$ belongs to the function j^{-1} . In I(4), replacement of x and y in (x, y) by numerals yields an element of L if and only if it transforms the formula $2x + 3y = 5$ into a valid connection of specific numbers; e.g., $(4, -1)$ is, and $(-1, 4)$ is not, an element of L . In a fourth way x and y are used in problems such as (11) and (12). These uses might be roughly described as *indicative*, *conjunctive*, *conditional*, and *imperative*.

II. *The Identity Function.* In the following statements, x designates the identity function, j , defined in the Introduction.

(1) The function $x+1$ is an extension of the (non-identical) functions

$$\frac{x^2 - 1}{x - 1} \quad \text{and} \quad \frac{x^2 - x - 2}{x - 2}.$$

(2) In the realm of integers mod. 2, the functions x and x^2 are identical. Either consists of the two pairs $(0, 0)$ and $(1, 1)$.

(3) The function $x^2 - 1$ is a polynomial, and $D(x^2 - 1) = 2x$.

$$(4) \frac{d \sin x}{dx} = \cos x.$$

(5) The class of all pairs (f, x, f) for any function f is an operator.

II(1), (2) are equivalent with I(1), (2), since the function $x+1$ is the class of all pairs $(x, x+1)$, and so on. Yet in none of the examples II is x a numerical variable. That the function $3+1$ is an extension of the function $\frac{3^x - 1}{3 - 1}$ is

utter nonsense, as is the expression $\frac{d \sin 3}{d3}$. That also in II(4) the letter x may be interpreted as the designation of the identity function is seen by defining for any two functions f and g the *rate of change* of f with regard to g as the function $\frac{df(x)}{dg(x)}$ (briefly, $\frac{df}{dg}$) assuming the value

$$\lim_{g(x) \rightarrow g(a)} \frac{f(x) - f(a)}{g(x) - g(a)}$$

for any number a for which this limit exists. Then $\frac{d \sin x}{dx}$ is the rate of change of the sine with the identity, $\frac{d \sin x}{dj(x)}$ (briefly, $\frac{d \sin}{dj}$).

III. *The Selector Functions.* In the following examples, x and y designate the 2-place selector functions, I and J , defined in the Introduction.

(1) If a function of x and a function of y are equal, then the two functions are constant; in symbols, if $f(x) = g(y)$, then there exists a constant function c such that $f = g = c$.

$$(2) \frac{\partial x^3 y^5}{\partial x} = 3x^2 y^5 \quad \text{and} \quad \frac{\partial x^3 y^5}{\partial y} = 5x^3 y^4.$$

(3) In the pure analytic geometry of the Cartesian plane, where points are defined as ordered pairs of numbers, and curves as certain classes of such pairs, $2x + 3y = 5$ and $2y + 3x = 5$ are different straight lines.

Clearly, (1) has the following meaning: If $f(I(a, b)) = g(J(a, b))$ for any a and b , then there exists a number c such that $f(t) = g(t) = c$ for any t . In (2), in contrast to I(10), the letters x and y cannot be replaced by numerals. That they may be interpreted as the selector functions is seen as follows. For any three 2-place functions F , G , and H , let $\left(\frac{\partial F}{\partial G}\right)_H$ denote the 2-place function assuming the value

$$\lim_{\substack{(x, y) \rightarrow (a, b) \\ H(x, y) = H(a, b)}} \frac{F(x, y) - F(a, b)}{G(x, y) - G(a, b)}$$

for any pair (a, b) for which this limit exists. If this function $\left(\frac{\partial F}{\partial G}\right)_H$ is called the *partial rate of change* of F with regard to G keeping H constant,

then the traditional expression $\frac{\partial F}{\partial x}$ designates the partial rate of change of F with regard to I keeping J constant; similarly, $\frac{\partial F}{\partial y}$ designates $\left(\frac{\partial F}{\partial J}\right)_I$.

(3) expresses the same fact as I(4), namely, that L and L' are different. But whereas, in the definition of L , x and y may be interchanged without affecting the meaning, in the traditional symbol for this line, as III(3) shows, x and y must not be interchanged. In the latter context they designate specific 2-place functions, in the former they are numerical variables.

IV. Real-Valued Complex Functions. In the following statements, x is the class of all pairs $(a+bi, a)$ for any a and b ; and y is the class of all pairs $(a+bi, b)$.

$$(1) z = x + yi.$$

$$(2) \frac{1}{3} \frac{dz^3}{dz} = (x + yi)^2.$$

Here, the letter z designates the identity function in the realm of complex numbers. (In the formula $D \sin z = \cos z$, it may be interpreted as this identity function or, in presence of the stipulation "for any z ," as a numerical variable whose scope is the class of all complex numbers.)

V. Indeterminates. The following formulae (without any qualifying or explanatory legends) belong to the theory of polynomial and rational forms.

$$(1) \frac{x^2 - 1}{x - 1} = \frac{x^2 - x - 2}{x - 2} = \frac{x + 1}{1}.$$

(2) In the realm of integers mod. 2, the polynomial forms $(x+1) \cdot (x-1)$ and $x^2 + 1$ are equal; the forms x and x^2 are considered as unequal.

$$(3) x^2 - 9y^2 = (x+3y) \cdot (x-3y).$$

It will be noted that the statements V(1), (2) about rational forms are (for opposite reasons) quite unparallel to the statements II(1), (2) about the corresponding rational functions. In contrast to a function, a form is not a class of pairs of numbers and thus is not meant to be evaluated for any specific argument. In contrast to a numerical variable, the letter x in a form is not supposed to be replaced by specific numerals. Just as a complex number may be considered as an ordered pair of real numbers, a polynomial form containing one letter is completely characterized by the sequence of its coefficients; that is to say, the form may be regarded as a sequence of numbers belonging to a given field. A rational form is an ordered pair of such sequences. Forms thus are hypercomplex numbers of a certain kind that are equated, added, and multiplied according to well-known laws. For instance, if, in a self-explanatory way, the rational forms in (1) are denoted by

$$\frac{(1, 0, -1)}{(1, -1)}, \quad \frac{(1, -1, -2)}{(1, -2)}, \quad \frac{(1, -1)}{(1)},$$

then the first two are equal because $(1, 0, -1) \cdot (1, -2) = (1, -1) \cdot (1, -1, -2)$. Operating with such sequences becomes more perspicuous if to each element of the sequence an indicator of its positions is appended. It is customary to suffix such an indicator as a quasi-exponent and to use as its quasi-base the letter x . This choice is motivated by the parallelism between rational forms and rational functions (the form x corresponding to the identity function x), even though this parallelism is very incomplete, as shown by the contrast between V(1), (2) and II(1), (2). In a form, each letter x thus is, as it were, a

holder of a place card (the quasi-exponent) describing the position of a coefficient. Such a letter in a form is called an indeterminate.

VI. Parts of Operational Symbols. The symbols $\frac{d}{dx}$, $\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial y}$ are often interpreted as synonyms of D , D_1 , and D_2 . In this case, the first x is a part of the symbol for the derivative just as D is a part of the synonymous symbol D —not more and not less. In the last two symbols, the letters x and y indicate which of two operations is intended: that pertaining to the first place in a 2-place function, or that pertaining to the second place. Of course, adopting this interpretation one attributes to the two letters x in the expression $\frac{d}{dx} \sin x$ totally discrepant meanings since the second x is either a numerical variable or the identity function, whereas in II(4) both x were interpreted as the identity function.

VII. Operators. The class mentioned in II(5) is often referred to as the operator x , for instance, in

$$(1) (D^2 + xD + 1) J_0 = 0.$$

Here J_0 is the 0-th Bessel function, and 1 designates the identity operator, that is, the class of all pairs (f, f) for any function f .

VIII. Specific Fluents (Abscissa and Ordinate). In contrast to pure analytic geometry, mentioned in III(3), the points and lines in a physical or postulational plane are not arithmetically defined. They are physical objects in the former (e.g., chalk dots and streaks on a blackboard), and undefined elements (called points and lines) satisfying certain assumptions, in the latter. In any physical or postulational plane, one may choose a Cartesian frame of reference consisting, essentially, of three non-collinear points, o , ξ , and η , called origin, first, and second unit point, respectively. Relative to such a frame, one can, by a well-known procedure, associate two numbers, which we denote by $x\pi$ and $y\pi$, with each point π in the plane. They are called the coordinates of π relative to the chosen frame. In particular,

$$x_o = y_o = y\xi = x\eta = 0 \quad \text{and} \quad x\xi = y\eta = 1.$$

In this way, two consistent classes of quantities are defined whose common domain is the class of all points in the plane under consideration:

the *abscissa*, x , is the class of all pairs $(\pi, x\pi)$ for any point π ;

the *ordinate*, y , is the class of all pairs $(\pi, y\pi)$ for any point π .

In defining these fluents, use has been made of a point variable: the letter π , which may be replaced by the designation of any specific point whereby one obtains a specific pair belonging to the fluent considered; e.g., replacing π by ξ and o in the definition of x one obtains the pairs $(\xi, x\xi) = (\xi, 1)$ and $(o, xo) = (o, 0)$ belonging to x . The latter pair belongs also to y .

In this sense x and y are used in the following statements in physical or postulational geometry.

(1) The lines $2x + 3y = 5$ and $2y + 3x = 5$ are different.

(2) If $y = \sin x$, then $\frac{dy}{dx} = \cos x$.

Here, $2x + 3y = 5$ and $y = \sin x$ designate the classes of all points π such that $2x\pi + 3y\pi = 5$ and $y\pi = \sin x\pi$, respectively. The rate of change of the fluent y with regard to the fluent x is defined as the fluent assuming the value

$$\lim_{x\rho \rightarrow x\pi} \frac{y\rho - y\pi}{x\rho - x\pi}$$

for any point π on the sine curve, ρ being confined to the same curve.

IX. Function Variables. In the following examples, y is a function variable put to various uses. Its scope in (1) and (2) is the class of all twice differentiable functions.

(1) If $D^2y + y = 0$, then y is a periodic function.

(2) Find y such that $D^2y + y = x$ and $y(0) = 1$.

The letter x in (2) as well as in $\frac{d^2y}{dx^2} + y = x$ designates the identity function; in $y''(x) + y(x) = x$ for any x , it is a numerical variable.

X. Fluent Variables. In the following examples, x and y are variables whose scopes are classes of fluents; that is to say, x and y are symbols that may be replaced by the designations of specific fluents.

(1) $\frac{d \sin x}{dx} = \cos x$ for any fluent x .

(2) If $y = \sin x$, then $\frac{dy}{dx} = \cos x$.

(3) If $y = \cos x$, then $\int_{x_0}^x y dx = \sin x - \sin x_{x_0}$.

In (2) and (3), x and y must not be replaced by numbers. For $\pi/2$ and 1 the antecedent in (2) is valid and yet the consequent is utterly nonsensical. However, one may replace x by the time t , and y by the position s of a linear oscillator, in which case (2) yields a statement about the rate of change of s with regard to t , namely,

if $s = \sin t$, then $\frac{ds}{dt} = \cos t$.

Here, t is defined as the class of all pairs (τ, t_τ) for any act τ of reading a clock, and s as the class of all pairs (σ, s_σ) for any act σ of reading the position of the oscillator, where t_τ and s_σ denote the numbers read as the results of the respective acts. If for any act τ of clock reading, $\Gamma\tau$ designates the *simultaneous* act of reading the position, then the rate of change of s with regard to t is defined as the fluent assuming for any act τ_0 the value

$$\lim_{t\tau \rightarrow t\tau_0} \frac{s(\Gamma\tau) - s(\Gamma\tau_0)}{t\tau - t\tau_0}.$$

In physics, this rate of change is identified with, or defined as, the *velocity* of the oscillator.

More generally, let x and y be any two fluents, defined as classes of pairs (α, x_α) and (β, y_β) , and let Γ be a pairing of an element $\Gamma\alpha$ belonging to the domain of y with any element α of the domain of x . One can introduce the rate of change of y with regard to x relative to the pairing Γ , namely, as the fluent $\frac{dy}{dx}$ assuming the value

$$\lim_{x_\alpha \rightarrow x_{\alpha_0}} \frac{y(\Gamma\alpha) - y(\Gamma\alpha_0)}{x_\alpha - x_{\alpha_0}}$$

for any α_0 in the domain of x for which this limit exists. In X(2) such a pairing was (as it is customary) tacitly taken for granted. In the example of the oscillator, the pairing was by *simultaneity*; in VIII(2), it was tacitly understood that with each point in the domain of x the same point in the domain of y should be associated. Also $\frac{df}{dg}$, as defined in II, subsumes under $\frac{dy}{dx}$ provided

that with each number x in the domain of g the same number $\Gamma x = x$ in the domain of f is paired (in other words, that $\Gamma = j$). With this understanding, X(1) yields II(4) if the fluent variable x is replaced by j .

Relative to such a pairing Γ , one may also define the *cumulation* of y with regard to x from α_0 to α , denoted by $\int_{\alpha_0}^{\alpha} y \, dx$, namely, as the number to which for various n and various sequences $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n = \alpha$ the product sums

$$y(\Gamma\alpha_0) \cdot (x\alpha_1 - x\alpha_0) + \dots + y(\Gamma\alpha_{n-1}) \cdot (x\alpha_n - x\alpha_{n-1})$$

are as close as he pleases provided that the largest of the numbers

$$|x\alpha_k - x\alpha_{k-1}| \quad (k = 1, \dots, n)$$

is sufficiently close to 0.

XI. *Sundries.* The meaning of x and y in formulae and expressions such as $dy = \sin x \, dx$ and $M(x, y) \, dx + N(x, y) \, dy$ will not here be discussed since the essential problem in these cases lies in the interpretation of the symbol d . Neither will be the numerous uses of x and y in vector algebra, vector analysis, etc. be elaborated on, since they are usually accompanied by stipulations that rule out the danger of confusion with the meanings discussed in this paper. The various meanings of x and y as so-called *random variables* are parallel to some of those herein listed and are summarized elsewhere [4].

XII. *Dummies.* In all previous examples, x and y have clearly defined, if totally discrepant, meanings. In the following formulae they have no meaning whatsoever.

$$(1) \int_0^1 \cos x \, dx = \sin 1 - \sin 0.$$

(2) $\int_a^{a+d} f(x) \, dx = \int_a^{a+d} f(y) \, dy$ for any two numbers a and d , and any function f that is continuous in the closed interval between a and $a+d$.

The simplest definition of the integral in (1) is

$$\int_0^1 \cos x \, dx = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \left[\cos \frac{1}{2n} + \cos \frac{3}{2n} + \dots + \cos \frac{2n-1}{2n} \right].$$

More generally,

$$\int_a^{a+d} f(x) \, dx = \lim_{n \rightarrow \infty} \frac{d}{n} \cdot \left[f\left(a + \frac{1}{2n} d\right) + f\left(a + \frac{3}{2n} d\right) + \dots + f\left(a + \frac{2n-1}{2n} d\right) \right].$$

Since the expression on the right side is free of x it is not surprising that there is a trend toward shedding the meaningless reference to x in the integral and toward writing simply $\int_a^{a+d} f$.

The English Language and the Mathematical Symbolism.

"[Old English] lacked an adequate system of pronouns and ambiguities were multiplied in Middle English when *hē* 'he', *hēo* 'she', and Anglian *hēo* 'they' . . . became identical in pronunciation. . . . The listener or reader [had to] gather the meaning from the context," says S. Potter in *Our Language*, and he continues: "That is why Middle English adopted and adapted these structural words ['they', 'them', and 'their'] from Scandinavian to supply its needs. Then, as now, intelligibility was a strong determining factor." "But although the *th*-forms must . . . be reckoned a great advantage to the language," says O. Jespersen in *Growth and Structure of the English Language*, "it took a long time before the old forms were finally displaced." But

eventually the new forms were generally adopted, for, as Potter puts it, "when men find that their words are imperfectly apprehended they naturally modify their speech and they deliberately prefer the unambiguous form."

In cases I and X of the traditional mathematical symbolism, the letter x , serving as a numerical and as a fluent variable, plays the roles of, as it were, mathematical "pronouns" that differ from one another as much as do "he" and "they". Moreover, in Cases II, III, IV, VII, and VIII, x has been found playing the roles of five (more or less unrelated) mathematical "nouns", in Cases V and VI the roles of (totally unrelated) suffixes, and in Case X no role whatsoever. Is it surprising that even excellent teachers of mathematics find "that their words are imperfectly apprehended" by some beginners? This often deplored situation is usually attributed to the profundity of mathematics, the rareness of mathematical talent in students, and the like, even though it might well be accounted for, and certainly is seriously aggravated by, the fact that ubiquitous mathematical symbols are ambiguous, nay, if the neologism be permitted, duodeciguous.

The physicist, operating with a number of fluents that exceeds the number of letters in the alphabet, is forced to use, in different contexts, the same letter in different meanings, e.g., t for the temperature as well as for the time; but no physicist has ever confused temperature and time. The equivocations in mathematics are more subtle and insidious. Indeed, which traditional treatises on analysis do clearly distinguish between the numerical variable in I(8), the identity function in II(4), and the fluent variable in X(1)?

The author has "naturally modified" mathematical expressions by introducing structural symbols that reflect the conceptual variety behind the letters x and y . Intelligibility has been a "strong determining factor" in the new approach to pure and applied analysis i.c. [1], where he "deliberately prefers unambiguous forms." The ten principal points of this clarification can be summarized as follows.

A. The designations of numbers (0, 1, 3, e, i, ...) and numerical variables (x , y , a , b , c , ...) are printed in roman type; the designations of specific functions (\log , \cos , the constant functions 0 , I , 3 , ...) and function variables in italics; operators in bold face. For instance, I(8) reads I(8') $D \sin x = \cos x$ for any x .

B. The following statements involving the specific functions j , I , and J are equivalent to the laws I(7)–(10) concerning any number belonging to a certain scope :

$$I(7') \quad \sin(2j) = 2 \sin j \cdot \cos j.$$

(The dot indicating multiplication must not be omitted. Mere juxtaposition means substitution, as in $\sin(2j)$.)

$$I(8'), (9') \quad D \sin = \cos \text{ and } D \log = j^{-1} \text{ (on the class of all numbers } > 0).$$

$$I(10') \quad \text{If } F = I^2 \cdot J^3, \text{ then } D_1 F = 3I^2 \cdot J^2 \text{ and } D_2 F = 5I^3 \cdot J^4.$$

Either of the following formulae, which are equivalent with I(5),

$$I(5') \quad I^2 - 9J^2 = (I + 3J) \cdot (I - 3J) \text{ and } J^2 - 9I^2 = (J + 3I) \cdot (J - 3I)$$

can be obtained from the other by substituting J for the first, and I in the second place. But the formulae I(5') are not identical in meaning.

C. The class of all x such that $x^2 - 1 = 0$ consists of all numbers for which the functions $j^2 - 1$ and 0 assume equal values. Since $j^2 - 1 = 0$, by itself, is a false statement, the class will be denoted by $\{j^2 - 1 = 0\}$. Similarly, L and L' in I(3) are $\{2I + 3J = 5\}$ and $\{2J + 3I = 5\}$, respectively. (Here, 5 is the constant 2-place function of value 5; it is denoted by $5^{(2)}$ where there is any danger of confusing it with the constant 1-place function 5 of value 5.) In pure Cartesian plane geometry, L and L' are called lines, and

$$I(4') \quad \{2I + 3J = 5\} \neq \{2J + 3I = 5\}.$$

D. The real-valued complex functions in IV might be denoted by re and im . In the realm of complex numbers

$$\text{IV(1')}, \text{ (2')} \quad j = re + i im \quad \text{and} \quad \frac{1}{2}Dj^2 = (re + i im)^2.$$

This is one of the many cases where intelligibility of the symbolism is achieved by following the way mathematicians talk. Who would orally refer to the x of a complex number rather than to its real part? Only in writing do mathematicians resort in this case as in so many others to the letters x and y that are so heavily fraught with connotations.

E. The italic letters x and y are reserved for specific fluents: Cartesian coordinates in a physical or postulational plane. The line consisting of all points π such that $2x\pi + 3y\pi = 5$ can, without the use of point variables, be written $\{2x + 3y = 5\}$. Clearly,

$$\text{VIII(5')} \quad \{2x + 3y = 5\} \neq \{2y + 3x = 5\}.$$

F. In view of the specific meaning of x and y , it is wise to avoid the use of x and y as function variables; and it is necessary to refrain from using x and y as fluent variables whose scopes include coordinates (just as one must avoid the use of e as a numerical variable whose scope includes the specific number e). The letters u, v, w, \dots and f, g, h, \dots are convenient fluent and function variables—symbols that may be replaced, e.g., by the time t and the ordinate y ; and the functions \cos and j^2 , respectively. The letters a, b, c, \dots are convenient as variables whose scopes consist of constant functions—symbols that may be replaced e.g., by the constant functions θ and 3 .

G. If g is a function, let c_g denote the constant operator of value g (that is, the class of all pairs (f, g) for any function f) just as, if b is any number, c_b (instead of b) might denote the constant function of value b (that is, the class of all pairs (x, b) for any number x).

Operators are objects of three operations (addition, multiplication, and substitution) just as are functions. The operators that are neutral with regard to these operations are c_θ , c_I , and j , that is, the classes of all pairs (f, θ) , (f, I) , and (f, f) for any function f , respectively. [5]

The operator defined in (II 5), which plays a great role in quantum mechanics, clearly is the product of c_j (the constant operator of value j) and the identity operator j , since

$$(c_j, j)f = c_j f \cdot j f = j \cdot f. \quad \text{Similarly, } (c_j \cdot D)f = c_j f \cdot Df = j \cdot Df.$$

Hence, VII (1) is rendered by

$$\text{VII (1')} \quad (D^{\text{II}} + c_j \cdot D + j)J_0 = 0 \quad \text{or} \quad (D^{\text{II}} + c_j \cdot D + j)J_0(x) = 0.$$

Roman type has been used for the number 0, italic type for the constant function θ of value 0. It will further be noted that roman numerals in the exponents distinguish substitutive iteration from multiplicative iteration which is traditionally expressed by arabic numerals in exponents. By the same token, one might distinguish the functions $\sin^{-1} = \arcsin$ and $\sin^{-1} = \csc$.

H. As a bearer of the ordinal number (as “basis of the exponent”) indicating the position of the coefficient in a polynomial or rational form, one may use an asterisk, $*$; in an n -ary form: $*_1, \dots, *_n$; in a binary form also: $*$ and \dagger . One has

$$\text{in arithmetic: } \frac{x^* - 1}{x - 1} = x + 1 \text{ for any number } x \neq 1;$$

$$\text{in the algebra of forms: } \frac{*_{\alpha} - 1}{* - 1} = * + 1;$$

$$\text{in analysis: } \frac{j^* - 1}{j - 1} = j + 1 \text{ (on the class of all numbers } \neq 1).$$

I. Since x and y mean abscissa and ordinate, $\frac{d}{dx}$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ mean the rates of change with regard to these specific fluents. More generally, the operators $\frac{d}{d}$ and $\frac{\partial}{\partial}$ associate a fluent with two fluents; the operators D , D_1 , and D_2 associate a function with one function.

J. Again following verbatim the mathematician's oral expression, one may denote the area from 0 to 1 under the cosine curve by $S_a^1 \cos$, just as one may write $S_a^{a+d} f$. Similarly, in presence of a symbol for the identity function, one may write $S_a^b \cos(2j)$ and $S_1^2 j^{-1}$. [6]. If a and b are two numbers, then $S_a^b f$ associates a number with any function f of a certain kind, whereas $\int_{\alpha_0}^x w du$ associates a number with two fluents u and w , if the domain of the former includes α_0 and x .

Clearly, instead of the symbols here presented, other notations might be used to differentiate visibly between meanings of x and y that are worlds apart. But it is believed that only the maintenance in some form of strict distinctions between those various meanings makes it possible to present algebra, analytic geometry, and pure as well as applied analysis as a system of formulae connected by articulate rules. [7]

K.M.

REFERENCES

1. This paper elaborates on ideas expressed in the Appendix to the author's book *Calculus. A Modern Approach*, Ginn and Co., Boston 1955.
2. Where there is no indication to the contrary, "number" in this paper means real number.
3. Traditionally, x and y are printed in italic type as are all single letters used as mathematical symbols.
4. Cf. the author's paper *Random Variables and the General Theory of Variables* to be published in the Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California, 1955.
5. Since substitution of operators (universally denoted by mere juxtaposition) is often called multiplication, the identity operator j is usually denoted by 1.
6. After submitting the present paper, I found that Dr. I. T. A. C. Adamson (*Mathematical Gazette*, September 1955) has suggested a similar symbol for definite integrals in his interesting review of my 1953 notes on calculus, edited in a greatly enlarged and improved form in the book i.e.¹.

1868. Dürer gained much by his visit to Venice. He had begun a translation of Euclid, and had been to Bologna to continue his study of mensuration. He had developed from grandiose structural compositions to a more thoughtful kind of painting.—Pierre Descargues, *Dürer*. [Per Mr. E. H. Lockwood].

1869. A STORM AT SEA.

Mr. Jolter was far from being unconcerned at the uncommon motion of the vessel, the singing of the wind, and the uproar he heard above him; . . . the poor governor's heart died within him, and he shivered with despair. His recollection forsaking him, he fell upon his knees in the bed, and fixing his eyes upon the book which was in his hand, began to pronounce aloud with great fervour, "The time of a complete oscillation in the cycloid is to the time in which a body would fall through the axis of the cycloid DV , as the circumference of a circle is to its diameter."—Smollett: *Peregrine Pickle*, Chap. 35—Saintsbury Ed., II, p. 2. (What was Mr. Jolter's book?) [Per Prof. H. G. Forder.]

SOME FUNDAMENTALS OF SPACE MECHANISMS

BY N. ROSENAUER

1. Plane Mechanisms. Since a link in a plane has three degrees of freedom the simplest closed plane mechanism with constrained motion must have four lower pairs, i.e. one more than the number of degrees of freedom. If the lower pairs are turning pairs, the mechanism is the well-known four-bar linkage. One or two turning pairs may be replaced by sliding pairs without affecting constrained motion, but not less than two turning pairs must remain in a closed link group [1].

2. Space Mechanisms. Similarly, since a link in space has six degrees of freedom, the simplest closed single-group space mechanism with constrained motion must have seven lower pairs i.e. one more than six.

If the lower pairs are turning pairs, the axes of the pairs generally do not intersect in space; an example of such a mechanism having seven links is shown in Fig. 1. This could be considered as the fundamental form of a closed single group space mechanism having binary links.

The same result has been obtained by H. Alt in another way [2].

3. Transformation of the Fundamental Space Mechanism. Six-link Space Mechanisms.

By combining two lower turning pairs into one cylindrical pair having two degrees of freedom, the number of links is reduced to six.

An example of such a mechanism is shown in Fig. 2, in which a cylindrical pair is placed between links 2 and 3. Denoting a pair having one degree of freedom as f_1 and a pair having two degrees of freedom as f_2 , we must consider f_2 as equivalent to $2f_1$. Hence the total number of degrees of freedom is now :

$$\Sigma f = 5f_1 + f_2 = 7f_1.$$

This shows that the number of degrees of freedom in all pairs remains seven.

4. Five-link Space Mechanisms. Repeating a similar combination of two turning pairs into a cylindrical pair twice, the number of links is reduced to five. An example of such a mechanism is shown in Fig. 3 where cylindrical pairs exist between links 2 and 3 as well as between 3 and 4.

The total number of degrees of freedom being :

$$\Sigma f = 3f_1 + 2f_2 = 7f_1.$$

Introducing in the mechanism of Fig. 1 a spherical joint which has three degrees of freedom and retaining the remaining four turning pairs with one degree of freedom each, we obtain another combination of a five-link space mechanism shown in Fig. 4.

The spherical joint which is placed between links 2 and 3, is denoted by f_3 and this is equivalent to $3f_1$.

The total number of degrees of freedom is :

$$\Sigma f = 4f_1 + f_3 = 7f_1.$$

5. Four-Link Space Mechanisms. Combining two turning pairs of the previous case into one cylindrical pair the number of links is again reduced by one giving a four-link space mechanism.

This mechanism is shown in Fig. 5 in which the spherical joint is placed between links 2 and 3; and the cylindrical pair between links 1 and 2. The total number of degrees of freedom is :

$$\Sigma f = 2f_1 + f_2 + f_3 = 7f_1.$$

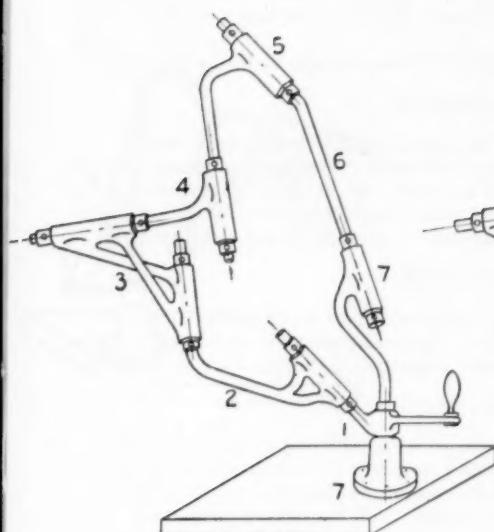


FIG. 1.

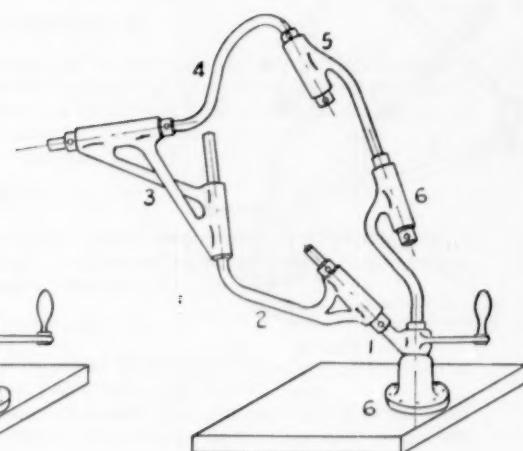


FIG. 2.

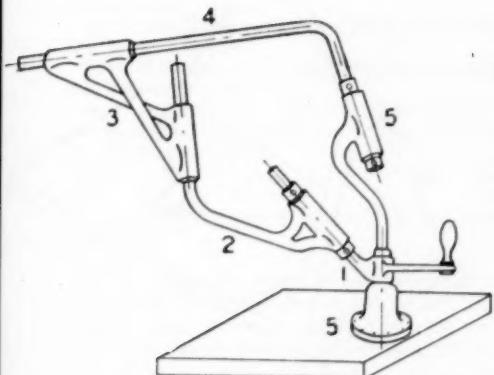


FIG. 3.

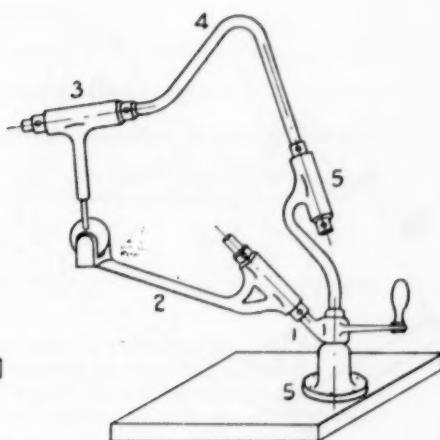


FIG. 4.

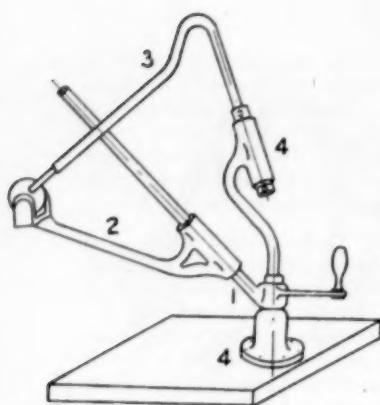


FIG. 5.

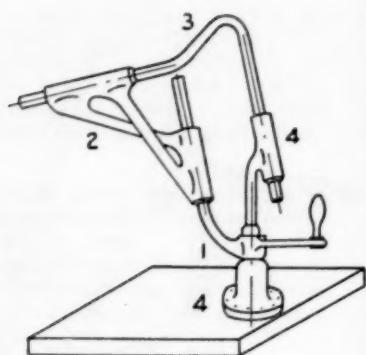


FIG. 6.

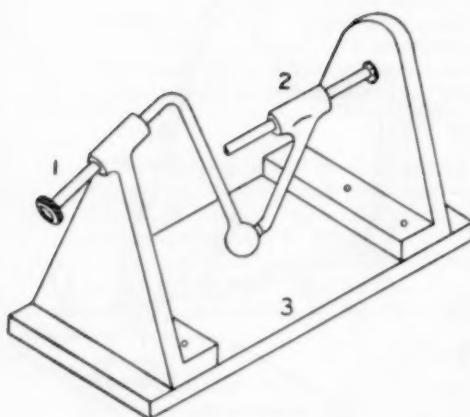


FIG. 7.

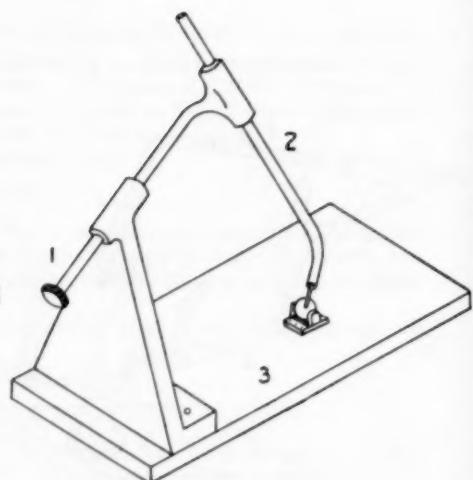


FIG. 8.

Another combination of a four-link space mechanism is developed from the mechanism of Fig. 3 by combining two turning pairs into one cylindrical pair ; this mechanism is shown in Fig. 6 in which there are three cylindrical pairs between links 1 and 2 ; 2 and 3 ; 3 and 4. The single turning pair is between links 4 and 1.

The total number of degrees of freedom is again :

$$\Sigma f = f_1 + 3f_2 = 7f_1.$$

6. Three-Link Space Mechanisms. If in the mechanism of Fig. 5 the two turning pairs are combined into one cylindrical pair, the number of links is again reduced by one i.e., we obtain a three-link space mechanism shown in Fig. 7 in which both cylindrical pairs are placed in the frame.

The total number of degrees of freedom is :

$$\Sigma f = 2f_2 + f_3 = 7f_1.$$

Another combination of a three-link space mechanism is shown in Fig. 8 where both the cylindrical pair and the spherical joint are placed in the frame. Again the total number of degrees of freedom is :

$$\Sigma f = 2f_2 + f_3 = 7f_1.$$

This paper shows possible combinations of single-group space mechanisms having different types of kinematic pairs, but only binary links.

Conditions for constrained motion of space mechanisms having links with more than two couplings and joints connecting more than two links are discussed by R. H. Macmillan [3].

Further examples of space mechanisms and the methods of constructing positions and velocities may be found in a publication by V. Dobrovolsky [4]. N.S.W. University of Technology,

N. R.

Sydney, Australia

REFERENCES

1. N. Rosenauer and A. H. Willis, *Kinematics of Mechanisms*, Sydney, (1953), pp. 271-278.
2. H. Alt. Die praktische Bedeutung der Raumgetriebe (1929) V. D. I. Vol. 73, No. 6, pp. 188-190.
3. R. H. Macmillan : The Freedom of Linkages. *Mathematical Gazette*, Vol. XXXIV, No. 307, Febr. 1950.
4. V. Dobrovolsky : *Method of Spherical Images in the Theory of Space Mechanisms*. (Russian) Publications of the Seminar of Theory of Machines and Mechanisms, Vol. 3, No. 11 Moscow (1947).

1870. Were the crew to slide forward at exactly the same pace as the boat is moving, so as to remain stationary in relation to the bank, then the total weight of the crew (say, five-sixths of the whole) would have been subtracted from the momentum.—*The Times*, 26 June, 1954. [Per Mr. R. F. Wheeler.]

1871. It has been recommended that Britain change to the Metric System. What nonsense is this? Britain and the Commonwealth were built by enterprise and industry founded on the bedrock of British weights and measures.

Our job is to lead the Continent—not follow it. The Metric System did not save France in 1940!—*Daily Mail*. [Per Mr. R. F. Wheeler.]

THE INVERTING TOP

D. G. PARKYN.

Introduction: The problem of the "tippe-top" has been discussed in several papers, one by Synge [1] makes the assumption of rolling and discovers instability when the axis is vertical with the peg up, but requires as a necessary condition that the top be not a solid of revolution. Since it appears that as far as is constructionally possible the top is axially symmetric, in which case the rolling motion is always stable, this solution would seem to be unrealistic. In a second group of papers Fokker [2] suggests from observations of peg traces that normal tops roll with no sliding, Braams [3] shows that for the tippe-top sliding will probably take place in the "rapid precession", and Hugenholtz [4] deduces general conditions in which such sliding will cause the peg of the top to fall. For the final rise on the peg these authors introduce a "rolling friction". Here results substantially in agreement with those of Hugenholtz are found from a simpler result by vector methods. It is further shown that the motion is such that during the final rise sliding must take place until the top is very nearly erect, and that the simple assumption of sliding friction is sufficient to explain the entire motion.

1. The Equations of Motion.

Let the mass of the top be M , radius " a ", axial and transverse moments of inertia C and A , $GC = h$ where G is the mass centre and C the centre of the spherical portion, GC being measured positive in the direction of the peg. Take unit vectors i , z along the axis and the upward vertical respectively. Further let r be the position vector of P , the point of contact, with respect to a fixed origin in the plane of the table, R the reaction at P and n the axial component of spin.

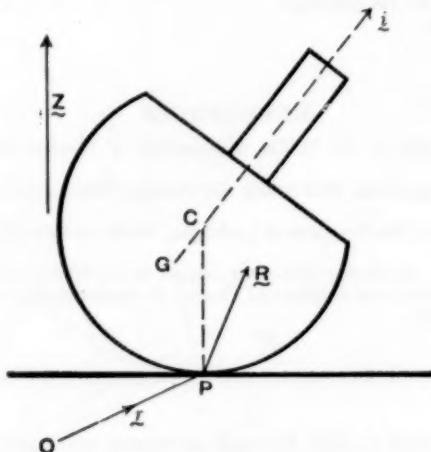


FIG. 1.

Then $\vec{PG} = az - hi$, and the equations of motion are

$$\frac{d}{dt} \left[Cni + Ai \wedge \frac{di}{dt} \right] = -(az - hi) \wedge R \quad \dots \dots \dots (1)$$

$$M \frac{d^2}{dt^2} \left[r + az - hi \right] = R - Mgz \quad \dots \dots \dots (2)$$

(cf. Milne—*Vectorial Mechanics*.)

The equations for the normal top are obtained by changing the sign of "h" and making it large compared with "a".

2. The Precession and Spin.

Eliminate \mathbf{R} :

$$\frac{d}{dt} \left[Cni + Ai \wedge \frac{di}{dt} \right] = -M(a\mathbf{z} - h\mathbf{i}) \wedge \left\{ g\mathbf{z} + \frac{d^2}{dt^2}(\mathbf{r} + a\mathbf{z} - h\mathbf{i}) \right\}.$$

Dot multiply by $a\mathbf{z} - h\mathbf{i}$, then

$$a\mathbf{z} \cdot \frac{d}{dt} \left(Cni + Ai \wedge \frac{di}{dt} \right) = hC \frac{dn}{dt},$$

whence $\mathbf{z} \cdot \left(Cni + Ai \wedge \frac{di}{dt} \right) = \frac{hC}{a} n + \text{const.}$

If ω be the angular velocity of precession about \mathbf{z} and $\mathbf{i} \cdot \mathbf{z} = \cos \alpha$, then this integral becomes

$$Cn \cos \alpha + A\omega \sin^2 \alpha = \frac{hC}{a} n + \text{const.}$$

Let N_0 be the spin when $\alpha = 0$, then

$$Cn(a \cos \alpha - h) + Aa \omega \sin^2 \alpha = CN_0(a - h) \quad \dots \dots \dots \quad (3)$$

and since $n \leq N_0$ we see that ω is essentially positive for $0 < \alpha < \pi$.

In absence of friction such a top can execute steady precession about the vertical, where ω can easily be shown to satisfy

$$A\omega^2 \cos \alpha - Cn\omega - Mgh = 0, \quad \dots \dots \dots \quad (4)$$

whence $\omega = Cn/A \cos \alpha$ or $-Mgh/Cn$ provided that n is large. Since ω is essentially positive it follows that we must choose the fast instead of the more usual slow precession of the erect top. The assumption of large n is not justified throughout the motion, but if we treat the frictional effect as a small perturbation, the top can only deviate slowly from the steady precession of a smooth top and equation (4) will be approximately valid throughout the gradual succession of "quasi-steady" precessions which form the motion. From (3) and (4) we obtain the result that

$$A\omega^2(a - h \cos \alpha) - CN_0\omega(a - h) - Mgh(a \cos \alpha - h) = 0,$$

whence, since N_0 is large, $\omega \approx \frac{CN_0(a - h)}{A(a - h \cos \alpha)}$.

Hence to the same approximation, from (3), $Cn \approx A\omega \cos \alpha$, and n actually changes sign during the motion and certainly does not remain large.

3. An Explicit Friction Assumption.

We consider the perturbing effect of sliding friction on a steady precession, and therefore since we assume the coefficient of friction small it is sufficient to take the value of the frictional component in the steady state. Thus, if \mathbf{v}_r is the relative velocity at the point of contact

$$\mathbf{v}_r = \frac{d\mathbf{r}}{dt} + a\mathbf{z} \wedge \left[n\mathbf{i} + \mathbf{i} \wedge \frac{di}{dt} \right]$$

and

$$\mathbf{R} = R_n \mathbf{z} - \mu R_n \frac{\mathbf{v}_r}{|\mathbf{v}_r|}$$

Hence if $\mathbf{i}_0, \mathbf{r}_0, n_0, \alpha$ and ω refer to a steady precession with $\mu = 0$, we may write correct to first order terms

$$\mathbf{R} = R_n \mathbf{z} - Mk \left\{ \frac{d\mathbf{r}_0}{dt} + a \mathbf{z} \wedge \left(n_0 \mathbf{i}_0 + \mathbf{i}_0 \wedge \frac{d\mathbf{i}_0}{dt} \right) \right\}$$

where $Mk = \frac{\mu R_n^3}{|\mathbf{v}_r^{(0)}|}$. From (2). \mathbf{z} , we find $\mathbf{R}_n = Mg - Mh \frac{d^2}{dt^2} (\mathbf{i} \cdot \mathbf{z})$ and equation (1) becomes on substitution

$$\begin{aligned} \frac{d}{dt} \left\{ Cn_0 \mathbf{i}_0 + A \mathbf{i}_0 \wedge \frac{d\mathbf{i}_0}{dt} \right\} &= -M(a \mathbf{z} - h \mathbf{i}_0) \wedge \left\{ \mathbf{z} \left[g - h \frac{d^2}{dt^2} (\mathbf{i} \cdot \mathbf{z}) \right] \right. \\ &\quad \left. - k \left[\frac{d\mathbf{r}_0}{dt} + a \mathbf{z} \wedge \left(n_0 \mathbf{i}_0 + \mathbf{i}_0 \wedge \frac{d\mathbf{i}_0}{dt} \right) \right] \right\} \dots\dots(5) \end{aligned}$$

valid for small k and i in the neighbourhood of \mathbf{i}_0 .

4. The Perturbation Equations.

Write $\mathbf{i} = \mathbf{i}_0 + \boldsymbol{\epsilon}$, $n = n_0 + \phi$, where $\boldsymbol{\epsilon}, \phi$ are assumed small and $\mathbf{i} \cdot \mathbf{z} = \cos \alpha$, $\frac{d\mathbf{i}_0}{dt} = \omega(\mathbf{z} \wedge \mathbf{i}_0)$. Further, since G is at rest in the steady precession, $\frac{d\mathbf{r}_0}{dt} = h \frac{d\mathbf{i}_0}{dt}$. Substitute in (5), retaining only first order terms, when after some reduction we obtain

$$\begin{aligned} Cn_0 \frac{d\boldsymbol{\epsilon}}{dt} + C\dot{\phi}\mathbf{i}_0 + C\phi\omega(\mathbf{z} \wedge \mathbf{i}_0) + A\mathbf{i}_0 \wedge \frac{d^2\boldsymbol{\epsilon}}{dt^2} + A\omega^2\boldsymbol{\epsilon} \wedge (\cos \alpha \mathbf{z} - \mathbf{i}_0) \\ + Mh^2(\mathbf{i}_0 \wedge \mathbf{z}) \frac{d^2}{dt^2} (\mathbf{z} \cdot \boldsymbol{\epsilon}) - Mgh(\boldsymbol{\epsilon} \wedge \mathbf{z}) = B[\mathbf{z}(a \cos \alpha - h) - \mathbf{i}_0(a - h \cos \alpha)] \dots(6) \end{aligned}$$

where $B = Mk[a n_0 + (h - a \cos \alpha)\omega]$.

(6). \mathbf{i}_0 ;

$$Cn_0 \mathbf{i}_0 \cdot \frac{d\boldsymbol{\epsilon}}{dt} + C\dot{\phi} + A\omega^2(\boldsymbol{\epsilon} \wedge \mathbf{z} \cdot \mathbf{i}_0) \cos \alpha - Mgh(\boldsymbol{\epsilon} \wedge \mathbf{z} \cdot \mathbf{i}_0) = -Ba \sin^2 \alpha.$$

But $\mathbf{i}_0 \cdot \boldsymbol{\epsilon} = 0$, whence $\mathbf{i}_0 \cdot \frac{d\boldsymbol{\epsilon}}{dt} = -\boldsymbol{\epsilon} \cdot \frac{d\mathbf{i}_0}{dt} = -\omega(\boldsymbol{\epsilon} \wedge \mathbf{z} \cdot \mathbf{i}_0)$ and on using (4) we have

$$C\dot{\phi} = -Ba \sin^2 \alpha,$$

whence, if $t = 0$ when the motion coincides with the steady precession,

$$C\phi = -Bat \sin^2 \alpha.$$

Substitute in (6) and change to a frame of reference rotating with \mathbf{i}_0 when $\frac{d\boldsymbol{\epsilon}}{dt} = \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \omega \mathbf{z} \wedge \boldsymbol{\epsilon}$, partial derivatives denoting the apparent rate of change.

$$\begin{aligned} Cn_0 \frac{\partial \boldsymbol{\epsilon}}{\partial t} + A\mathbf{i}_0 \wedge \frac{\partial^2 \boldsymbol{\epsilon}}{\partial t^2} - 2A\omega \cos \alpha \frac{\partial \boldsymbol{\epsilon}}{\partial t} + A\omega^2(\mathbf{z} \cdot \boldsymbol{\epsilon})(\mathbf{i}_0 \wedge \mathbf{z}) + Mh^2(\mathbf{i}_0 \wedge \mathbf{z}) \frac{\partial^2(\boldsymbol{\epsilon} \cdot \mathbf{z})}{\partial t^2} \\ = B(\mathbf{i}_0 \cos \alpha(h - a \cos \alpha) + \mathbf{z}(a \cos \alpha - h) + a \sin^2 \alpha(\mathbf{z} \wedge \mathbf{i}_0)\omega t). \end{aligned}$$

$$\begin{aligned} &\wedge \mathbf{i}_0; \\ A \frac{\partial^2 \boldsymbol{\epsilon}}{\partial t^2} - (Cn_0 - 2A\omega \cos \alpha)\mathbf{i}_0 \wedge \frac{\partial \boldsymbol{\epsilon}}{\partial t} + A\omega^2(\mathbf{z} \cdot \boldsymbol{\epsilon})(\mathbf{z} - \cos \alpha \mathbf{i}_0) + Mh^2[\mathbf{z} - \cos \alpha \mathbf{i}_0] \frac{\partial^2(\boldsymbol{\epsilon} \cdot \mathbf{z})}{\partial t^2} \\ &= B[(\cos \alpha \dot{\mathbf{i}}_0 - \mathbf{z})a\omega t \sin^2 \alpha + (\mathbf{z} \wedge \mathbf{i}_0)(a \cos \alpha - h)]. \end{aligned}$$

Introduce vectors \mathbf{j}, \mathbf{k} in the moving frame such that $\mathbf{i}_0, \mathbf{j}, \mathbf{k}$ form a positive

triad and \mathbf{k} is in the direction $\mathbf{z} \wedge \mathbf{i}_0$, when $\mathbf{z} = \cos \alpha \mathbf{i}_0 - \sin \alpha \mathbf{j}$. The above equation then reduces to

$$A \frac{\partial^2 \mathbf{e}}{\partial t^2} - (Cn_0 - 2A\omega \cos \alpha) \mathbf{i}_0 \wedge \frac{\partial \mathbf{e}}{\partial t} + A\omega^2 \sin^2 \alpha (\mathbf{j} \cdot \mathbf{e}) \mathbf{j} + Mh^2 \sin^2 \alpha \mathbf{j} \frac{\partial^2 (\mathbf{e} \cdot \mathbf{j})}{\partial t^2} = B \{ a\omega t \sin^2 \alpha \mathbf{j} + (a \cos \alpha - h) \sin \alpha \mathbf{k} \} \quad \dots \dots \dots (7)$$

Set $\epsilon = \lambda j + \mu k$, substitute in (7), when on taking components we obtain

$$(A + Mh^2 \sin^2 \alpha) \ddot{\lambda} + (Cn_0 - 2A\omega \cos \alpha) \dot{\mu} + A\omega^2 \sin^2 \alpha \lambda = Ba\omega t \sin^3 \alpha$$

$$A \ddot{\mu} - (Cn_0 - 2A\omega \cos \alpha) \dot{\lambda} = B(a \cos \alpha - h) \sin \alpha.$$

If we assume $\dot{\mu} = 0$ when $t = 0$, then

$$A\dot{\mu} - (Cn_0 - 2A\omega \cos \alpha)\lambda = Bt \sin \alpha (\alpha \cos \alpha - h),$$

and

$$A(A + Mh^2 \sin^2 \alpha)\dot{\lambda} + ((Cn_0 - 2A\omega \cos \alpha)^2 + A^2 \omega^2 \sin^2 \alpha)\lambda = Bt \sin \alpha (Aa\omega \sin^2 \alpha - (Cn_0 - 2A\omega \cos \alpha)(a \cos \alpha - h))$$

Thus λ has a periodic part indicating the stability of the frictionless motion together with a forced motion given by the particular integral

$$\lambda = \frac{Bt \sin \alpha [Aa\omega \sin^2 \alpha - (Cn_0 - 2A\omega \cos \alpha)(a \cos \alpha - h)]}{[(Cn_0 - 2A\omega \cos \alpha)^2 + A^2\omega^2 \sin^2 \alpha]},$$

or, substituting the approximate value of $n_0 = \frac{A}{C} \omega \cos \alpha$,

$$\lambda = \frac{Mk \sin \alpha}{A} \left[h + \left(\frac{A - C}{C} \right) a \cos \alpha \right] (a - h \cos \alpha) t.$$

5. Deductions.

It is convenient to introduce the notation $(A - C)/C = \beta$, $h/a = \gamma$, when the condition shows that the top axis will rise or fall as

$$(\gamma + \beta \cos \alpha)(1 - \gamma \cos \alpha) < 0 \text{ or } > 0.$$

Thus, for the tippe-top where $0 < \gamma < 1$, the peg will fall when

$$(\gamma + \beta \cos \alpha) > 0.$$

We may distinguish the following types:

(i) $|\beta| < \gamma$ allows complete inversion.

(ii) $\beta > \gamma$ implies that the process will terminate when $\cos \alpha = -\gamma/\beta$, a result which will not affect the tippe-top, since the peg will come into contact with the ground before this occurs unless β is much greater than γ , that is A large compared with C .

(iii) $-\beta > \gamma$ implies that falling cannot begin until $\alpha > \cos^{-1} |\gamma| / |\beta|$. While not precisely realised in the normal models this condition does have an effect. The spherical surface is flattened at the "pole"—presumably to give increased statical stability—giving a locally small value of γ on the region of greater curvature between the flattened and normal spherical portion of the surface. Thus the peg begins to fall until the "ridge" is reached, there carrying out a semi-stable precession for quite a long time. The peg may even be observed to jerk upwards and regain its position several times before a chance unevenness of the spinning surface carries the top beyond this region. The effect

will not, however, be noticed unless the top is initially spun accurately in the vertical, as a normal "hand spin" gives sufficient deviation to take the top out of the flattened region.

6. Rising on the Peg.

The impulsive force when the peg comes into contact with the spinning surface will be small and may be ignored. The effect of contact of both peg and spherical surface, with sliding at both points, is to add a second term of the same type to the R.H.S. of equation (6), so that the previous analysis holds qualitatively and there will be a torque tending to increase the reaction on the peg until the spherical portion leaves the spinning surface completely. Hereafter the motion is of the same type as before but with different values of "a" and "h".

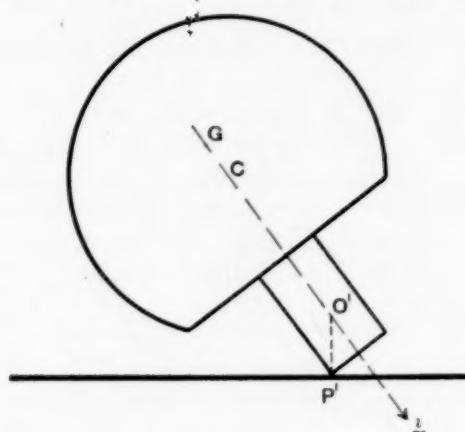


FIG. 2.

Let "c" be the radius of the peg and $GA = H$. To use the previous analysis write

$$\gamma' = \frac{GO'}{P'O'} = \left(\frac{H}{c} \sin \alpha + \cos \alpha \right),$$

when the condition for rising, that is $\dot{\theta}$ still falling, is effectively

$$\frac{H^2}{c^2} (-\cos \alpha) > 0$$

since H/c is large compared with β . Since $\pi/2 < \alpha \leq \pi$ this condition is always satisfied.

Assuming the motion to be a gradual succession of steady precessions, so that G moves effectively in a vertical line, the relative velocity at P' is given by

$$|\mathbf{v}_r| = \omega H \sin \alpha + nc,$$

and since the relation $Cn = A\omega \cos \alpha$ remains approximately valid throughout the motion, the only assumption being that the change of steady precessions is gradual, we have

$$|\mathbf{v}_r| \approx \omega \left(H \sin \alpha + \frac{A}{C} c \cos \alpha \right).$$

Thus the relative velocity vanishes when $\tan \alpha = -\frac{A}{C} \cdot \frac{c}{H}$, or when the inclination to the vertical is about 8° , if $A \sim C$ and $H = 8c$. Certainly sliding maintains until the top is effectively erect and no assumption of rolling friction is required to cover this stage of the motion. It appears then that the single assumption of sliding friction is sufficient to explain at least qualitatively all phases of the motion of the tippe-top. The problem of the rise of a normal top in slow precession on a blunt peg is interesting, and may well require "rolling friction", but the effect would seem to be irrelevant here when complete erection on to the flat-ended peg never takes place.

University of Natal.

D. G. P.

REFERENCES

1. Synge, J. L., *Philosophical Magazine*, 43/724/1952.
2. Fokker, A. D., *Physica*, 18/489/1952.
3. Braams, C. M., *Physica*, 18/503/1952.
4. Hugenholtz, N. M., *Physica*, 18/515/1952.

SUMMER COURSES IN ITALY

The Centro Internazionale Matematico Estivo (C.I.M.E.) on the advice of Professor E. Bompiani arranges each year summer courses in various branches of mathematics. These courses last ten days and usually take place in the Villa Monastero, Varenna (Lago di Como), but this year courses were also held at the Fondazione G. Cini, Isola S. Giorgia, Venezia, and the University of Pavia.

At each course three mathematicians of international standing are invited to give eight one hour lectures each. In addition to this there are seminars on special topics. The lectures are given in Italian, French, or English, and are designed to introduce the mathematically trained listener to a new field, as well as to provide the specialist with a survey of recent research. This year there were courses on the following five subjects : the Riemann Roch theorem and connected questions, analytic theory of numbers, topology, recent results in Elasticity and aero-dynamics, and projective differential geometry with particular regard to algebraic concepts.

The number of students attending a conference is limited to forty. Because of this small number there is a very friendly and informal atmosphere, and the young mathematicians derive great benefit from their numerous and endless discussions with each other and with the lecturers. Most of the students (each of whom receives a generous contribution to his costs from the C.I.M.E.) are young professors of Italian universities, but there is also a sprinkling of mathematicians of various European countries, as well as of teachers of Italian secondary schools who are anxious not to lose touch with the subject as it is taught at universities.

E. STEIN

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*.

DEAR SIR,

Simple Subtraction.

I think the admirable report on Mathematics in Primary Schools might be improved by mentioning an explanation of subtraction which I learnt from a boy who was 6 years old.

In a sum such as

$$\begin{array}{r} 67 \\ - 29 \\ \hline 38 \end{array}$$

using bundles of matches (tied in tens) he had been taught to borrow ten, so that he had 5 tens and 17 units. He then found the 8 and was taught to go on 2 from 5. I tried to persuade him to take 3 (i.e. 1 + 2) from the 6. Suddenly he said "Oh! I see, you take away the 1 and the 2 at the same time."

Soon after Dr. Ballard's *Teaching the Essentials of Arithmetic* came out, I told him of the above. He wrote me a most enthusiastic letter, saying that it was the best explanation he had ever seen and he should always use it in future

Yours, etc., A. W. SIDDONS

To the Editor of the *Mathematical Gazette*.*Query.*

DEAR SIR,—The alleged inequality

$$f(x_1, \dots, x_n) = \frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \dots + \frac{x_{n-1}}{x_{n-1}+x_n} + \frac{x_n}{x_n+x_1} \geq \frac{1}{2}n,$$

where $x_r > 0$, $r=1$ to n , given by H. S. Shapiro (*American Math. Monthly*, Oct. 1954) is true when $n=3$ and $n=4$.

The following example, due to Professor Lighthill, shows that it is not true in general :

When $n=20$, take x_1, x_2, \dots, x_{20} to be (in that order) $1+5\epsilon, 6\epsilon, 1+4\epsilon, 5\epsilon, 1+3\epsilon, 4\epsilon, 1+2\epsilon, 3\epsilon, 1+\epsilon, 2\epsilon, 1+2\epsilon, \epsilon, 1+3\epsilon, 2\epsilon, 1+4\epsilon, 3\epsilon, 1+5\epsilon, 4\epsilon, 1+6\epsilon, 5\epsilon$, where ϵ is small and positive ; then it is easy to see that

$$f(x_1, \dots, x_n) = 10 - \epsilon^2 + O(\epsilon^2).$$

Are there simple examples which show that the inequality is untrue (i) if n is odd, (ii) if $4 < n < 20$?

Are there values of n greater than 4 for which the inequality is true?

Yours etc., C. V. DURELL

MATHEMATICAL NOTES

2631. Zips.

Having a function $f(x)$ defined in a given interval $[ab]$ and a number n let us consider a series

$$(1) \quad z_N = \sum_{r=0}^N (-1)^r \binom{n}{r} f(b - rh) h^{-n},$$

where

$$(2) \quad h = (b - a)/N, \text{ and } \binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}.$$

If (1) has a limit when $N \rightarrow \infty$ we may denote it by

$$(3) \quad {}_a^b Zf = \lim_{N \rightarrow \infty} \sum_{r=0}^N (-1)^r \binom{n}{r} f(b - rh) h^{-n}$$

and call it zip of the rank n of the function f in the interval $[ab]$.

For any other function $g(x)$ we have

$$(4) \quad {}_a^b Z(f+g) = {}_a^b Zf + {}_a^b Zg,$$

and for any constant c

$$(5) \quad {}_a^b Zcf = c {}_a^b Zf.$$

These are possibly the only general statements following straight from the definition (3). But there are some other particular points of interest. E.g.

$$(6) \quad {}_a^b Zf = f(b).$$

$$(7) \quad {}_a^b Zf = \int_a^b f(x) dx.$$

Further if for a positive integer n the function f possesses an n th derivative in $[ab]$ then we have

$$(8) \quad {}_a^b Zf = f^{(n)}(b).$$

For in fact for any positive integer n (3) becomes

$$(9) \quad {}_a^b Zf = \lim_{\substack{r=0 \\ h \rightarrow 0}} \frac{\sum_{r=0}^n (-1)^r \binom{n}{r} f(b - rh)}{h^n}.$$

Applying to (9) n times consecutively the rule of l'Hospital we get

$$(10) \quad {}_a^b Zf = \lim_{\substack{h \rightarrow 0}} \frac{(-1)^n \sum_{r=0}^n (-1)^r \binom{n}{r} r^n f^{(n)}(b - rh)}{n!}.$$

Now $(-1)^n \sum_{r=0}^n (-1)^r \binom{n}{r} r^n$ is a G. Boole's number and is equal to $n!$, hence (8).

To avoid reference to G. Boole's *A Treatise on the Calculus of Finite Differences* I am going to prove the statement in question independently by Mathematical Induction in conjunction with another statement—

$$(11) \quad \sum_{r=0}^n (-1)^r \binom{n}{r} r^s = 0, \text{ for all } s < n,$$

$$\sum_{r=0}^n (-1)^r \binom{n}{r} r^n = (-1)^n n!$$

s, n positive integers.

Let us denote by S the set of integers n for which (11) is true. It is easy to ascertain that

I : The number 1 belongs to S .

II : If k belongs to S then $(k+1)$ also belongs to S .

$$\begin{aligned} \text{For } \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} r^s &= -(k+1) \sum_{r=1}^{k+1} (-1)^{r-1} \binom{k}{r-1} r^{s-1} \\ &= -(k+1) \sum_{\lambda=0}^k (-1)^\lambda \binom{k}{\lambda} (\lambda+1)^{s-1} \\ &= -(k+1) \sum_{\lambda=0}^k (-1)^\lambda \binom{k}{\lambda} \sum_{\mu=0}^{s-1} \binom{s-1}{\mu} \lambda^{s-1-\mu}; \end{aligned}$$

so that, changing the order of summation in the last line, we obtain

$$(12) \quad \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} r^s = -(k+1) \sum_{\mu=0}^{s-1} \binom{s-1}{\mu} \sum_{\lambda=0}^k (-1)^\lambda \binom{k}{\lambda} \lambda^{s-1-\mu}$$

and since (11) is true for k the right side of (12) is zero for all $s \leq k$ while for $s = k+1$ it becomes $(-1)^{k+1}(k+1)!$.

III : Induction warrants that S includes all positive integers.

(7) and (8) prove that integration and differentiation are embraced in the formula (3) which possesses a definite meaning for any real number n . But it is impossible to interpret zip of a function in terms of derivatives and integrals when n is not an integer.

J. MASSALSKI

2632. A note on numerical integration.

Suppose that $f(x)$ is an arbitrary polynomial or alternatively that $f(x)$ can be expanded in a Taylor series convergent in $-1 \leq x \leq 1$.

Then if

$$f(x) = \sum_{r=0}^{\infty} a_r x^r$$

$$\int_{-1}^1 f(x) dx = 2 \sum_{r=0}^{\infty} \frac{a_{2r}}{2r+1}. \quad (1)$$

We wish to approximate to this integral by an expression

$$A = \sum_{r=1}^n \lambda_r f(\alpha_r) \quad . \quad (2)$$

If we choose $\sum_{r=0}^n \lambda_r x_r^{2k} = \frac{2}{2k+1}, \quad k=0, 1, 2, \dots, p-1$ (3)

then A will give the exact value of (1) if $f(x)$ is a polynomial of degree $\leq 2p - 1$. It will give an approximation to this integral if $f(x)$ is of a higher degree or if $f(x)$ is an infinite series.

We consider equations (3) and (4). We observe that the equations (4) are satisfied if we choose the values of α in pairs, so that if α_r is one value of α then $-\alpha_r$ is another value, the corresponding values of λ being equated. We may in that case write (2) as

and the equation (3) as

In order that (2) should give the exact value of the integral for a polynomial of degree $2p - 1$, we have to find values of α and λ to satisfy p equations (6). Various methods suggest themselves in the choice of suitable values of α and p .

Method (1). We choose p values of α and use these equations to find the corresponding values of λ . For a polynomial of degree 3 we have to choose 2 values of α . We choose $\alpha_1 = 0$ and $\alpha_2 = 1$. We readily get $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$ and

$$A = \frac{1}{3} \{4f(0) + f(1) + f(-1)\}.$$

This is Simpson's rule.

For a polynomial of degree 5 we choose $\alpha_1 = 0$, $\alpha_2 = 1$ and for α_3 we might choose $\frac{1}{2}$. We then get $\lambda_1 = -\frac{3}{2}$, $\lambda_2 = \frac{7}{4}$, $\lambda_3 = \frac{33}{8}$ and so

$$A = \frac{1}{45} \{ 12f(0) + 7(f(1) + f(-1)) + 32(f(\tfrac{1}{2}) + f(-\tfrac{1}{2})) \}$$

Method (2). We might choose the values of λ and use the equations to determine the corresponding value of α . We might for example choose the λ 's so that the value of the integral is the range multiplied by the arithmetic mean of the values of $f(x)$. For a cubic since there are just two equations to satisfy, $\lambda = 1$ and $\alpha_3^2 = \frac{1}{3}$ and so

$$A = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right).$$

This is Tchebychef's formula.

For a polynomial of degree 5, $\lambda_1 = \lambda_2 = \frac{1}{2}$ and the values of α are given by

$$\alpha_1^{\pm} = \frac{1}{2} \left(1 \pm \frac{2}{\sqrt{5}} \right), \quad \alpha_3^{\pm} = \frac{1}{2} \left(1 - \frac{2}{\sqrt{5}} \right).$$

Method (3). We might use the p equations which have to be satisfied for a polynomial of degree $2p-1$ to determine the values of α as well as the corresponding values of λ . To get the exact value for a polynomial of degree $2p-1$ we have to satisfy p equations. If p is even we use the equations to determine $p/2$ values of α and the corresponding values of λ . If p is odd we

choose $\alpha_1 = 0$ and use the equations to determine $\frac{p-1}{2}$ values of α and the $\frac{p+1}{2}$ corresponding values of λ . It will be observed that in both cases p values of $f(x)$ have to be used. This is the minimum number of values of $f(x)$ that can enter into the evaluation of the exact value of the integral of a polynomial of degree $2p-1$. For a cubic $p=2$, one value of α is required and one value of λ . We get Tchebychef's formula again. For a polynomial of degree 5, $p=3$ and so is odd.

$$\text{I get } \alpha_1 = 0, \alpha_2 = \sqrt{\frac{2}{3}} \text{ and } A = \frac{1}{6} \{ 8f(0) + 5(f\sqrt{\frac{2}{3}}) + f(-\sqrt{\frac{2}{3}}) \}.$$

For a polynomial of degree 7.

$$\text{I get } \alpha_1^2 = \frac{1}{7}(3 + 2\sqrt{\frac{2}{7}}), \alpha_2^2 = \frac{1}{7}(3 - \sqrt{\frac{2}{7}})$$

$$\lambda_1 = \frac{1}{2} - \frac{1}{4}\sqrt{\frac{2}{7}}, \lambda_2 = \frac{1}{2} + \frac{1}{4}\sqrt{\frac{2}{7}}.$$

Method (4). We might choose some of the values of α and use the equations to determine the remaining values and all the corresponding values of λ . The simplest case in which this method might be usefully employed is for a polynomial of degree 7. If we choose $\alpha_1 = 0, \alpha_2 = 1$, and use the four equations for a polynomial of degree 7 to determine α_3 and the three values of λ we get $\alpha_3 = \sqrt{\frac{2}{7}}, \lambda_1 = \frac{1}{4}\sqrt{\frac{2}{7}}, \lambda_2 = \frac{1}{16}, \lambda_3 = \frac{4}{9}\sqrt{\frac{2}{7}}$ and so

$$A = \frac{1}{60} (64f(0) + 9(f(1) + f(-1)) + 49(f(\sqrt{\frac{2}{7}}) + f(-\sqrt{\frac{2}{7}})).$$

In note 2531, S. J. Tupper, working along different lines obtains a formula of type (5) for a polynomial of degree 5. His formula may be obtained by choosing $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = \sqrt{\frac{2}{3}}$

It leads to the result

$$A = \frac{1}{15} \left\{ 8f(0) + 2(f(1) + f(-1)) + 9 \left(f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\sqrt{\frac{1}{3}}\right) \right) \right\}.$$

It is less accurate than the formula just obtained but has the advantage that the coefficients are smaller. If we wish to avoid "difficult" numbers like $\frac{1}{\sqrt{3}}$ or $\sqrt{\frac{3}{7}}$ we might take $\alpha_3 = \frac{2}{3}$, since $\frac{2}{3}$ is approximately $\sqrt{\frac{3}{7}}$.

We get

$$A = \frac{1}{180} [110f(0) + 14f(1) + f(-1) + 81 \{ f(\frac{2}{3}) + f(-\frac{2}{3}) \}].$$

This is somewhat more accurate for a polynomial of degree 7, than the formula of S. J. Tupper.

University of Natal.

P. STEIN

2633. Units and Dimensions.

The purpose of this note is to take issue with the treatment of this subject in an article with this title by H. V. Lowry, in the issue for Sept. 1954.

It is possible to treat a physical equation, such as $F=ma$, by two different methods. In Method I, the "physical method", the symbols are held to represent the physical entities force, mass and acceleration, regardless of the units in which they are measured. When one comes to "put the numbers in", it is necessary to put in not only numbers but also the units in which they are measured. Thus for F we may write 1 lb-wt, 32.2 pdl, or 454 g-wt at will. One can separate the entity represented by a symbol into two elements, a number and a unit, and dimensional analysis is concerned with the relations which must hold between the units. In Method II, the "numerical

method ", the symbols are held to represent only the numbers which measure the force etc., in some predetermined system of units. The equation is consequently only valid in that system of units, and when a change of units is made it may have to be modified by the insertion of a numerical conversion factor depending on the units used ; such a factor is referred to below as a unit-conversion factor. Since the equation only concerns numbers no question of dimensions arises and it is not legitimate to use dimensional analysis.

In Mr. Lowry's paper no clear distinction is made between the use of Methods I and II, and for this reason errors arise. In paragraph 1, he is using method II, since the equation is written with a unit-conversion factor k^{-1} in an attempt to combine this with dimensional analysis he is led into confusion. In paragraph 2, on the other hand, he explicitly introduces a symbol for the velocity of light (and not its numerical measure) into equations (iii) and (iv), and must therefore be using Method I, although at the same time he introduces a series of γ 's, which are unit-conversion factors only appropriate to Method II.

The confusion in Mr. Lowry's paragraph 1 may be illustrated by noting that we are told that " k is a dimensionless number ". Now the expression derived for k contains factors which are clearly ratios—the numbers of ounces in a ton, of feet in a mile, and of seconds in an hour—but in addition to these it contains the factor 32 for the gravitational acceleration, g . Is g dimensionless? The truth is that dimensions are meaningless when, as here, Method II is being used.

The reader may be interested to compare the alternative working by Method I. As this method is, perhaps, less familiar it should be explained that the units in which a quantity is to be measured can be specified by using a solidus, which is read aloud by the word " in ". Thus if F is a force, regarded as a physical entity, then $F/\text{lb-wt}$, read " F in lb-wt " is the measure of that force in the specified unit. The solidus can be treated as an ordinary symbol of division, and units can be cross-multiplied, cancelled, etc., in the ordinary way. To change units, multiply by unity factors such as 1 ft/12 in. The solution of Mr. Lowry's problem by Method I runs as follows.

Start from the physical law :

$$F = ma,$$

$$\therefore \frac{F}{\text{ton-wt}} = \frac{m}{\text{oz mile/h}^2} \frac{a}{\text{oz mile/h}^2} \frac{\text{oz mile/h}^2}{\text{ton-wt}}.$$

On multiplying by suitable unity factors

$$\begin{aligned} \frac{F}{\text{ton-wt}} &= \frac{m}{\text{oz mile/h}^2} \frac{a}{\text{ton-wt}} \frac{\text{oz mile}}{\text{h}^2} \frac{1 \text{ ton}}{16 \times 2240 \text{ oz}} \\ &\quad \times \frac{1 \text{ ton-wt}}{1 \text{ ton } 32.2 \text{ ft/s}^2} \frac{5280 \text{ ft}}{1 \text{ mile}} \frac{1 \text{ h}^2}{3600^2 \text{ s}^2}. \end{aligned}$$

On cancelling units and collecting factors

$$\frac{F}{\text{ton-wt}} = \frac{m}{\text{oz mile/h}^2} \frac{a}{16 \cdot 2240 \cdot 32.2 \cdot 3600^2}$$

Consider next Mr. Lowry's paragraph 2 on electrical units. Since the whole argument is a dimensional one Method II is inadmissible and Method I must be used. This implies that the γ 's must be omitted, as being unit-conversion factors. On p. 182 it is stated that " K, μ are to be taken as 1 in a vacuum and so to be independent of the unit system ". In fact, neither K nor μ is

taken to be 1 in the MKS system, but assigning arbitrary values, whether 1 or not, does not make K or μ non-dimensional. It is from this fallacy that Mr. Lowry derives his erroneous statement that no more than three dimensions are required in electro-magnetic problems. The fallacy is easily demonstrated by considering the (unrationalized) gravitational equation

$$F = \frac{GM^2}{r^2},$$

which corresponds with the unrationalized magnetic equation

$$F = \frac{m^2}{\mu r^2}.$$

The dimensions of F , M and r are respectively MLT^{-2} , M and L , so that $G = Fr^2/M^2$ has dimensions $M^{-1}L^3T^{-2}$. We can assign an arbitrary value to G , but we cannot make it non-dimensional. We cannot, however, assign a value to G independently of the units of mass, length and time; it does not, therefore represent an additional dimension. Turning now to the magnetic equation, it is found that, in contrast to the gravitational case, even after fixing units of mass, length and time, it is possible to take an arbitrary value for μ . This implies that there is a fourth fundamental dimension in electro-magnetic problems, since a whole range of unitary systems exist, having the same measures of mass, length and time, but different values of μ . There is no compulsion to choose μ as the fourth dimension, and in the MKS system, which is really the MKSA system, the ampere has been chosen as the fourth fundamental unit. It follows that the dimensions of μ , for instance, are $MLT^{-2}I^{-2}$.

Provided that correct expressions are used for the dimensions there is no difficulty in checking dimensions whichever system of units is being used. Physical equations must be dimensionally consistent whatever units are to be used when "the numbers are put in". Because mass infrequently enters into electro-magnetic problems it is often more convenient in checking dimensions to take as dimensions potential, current, length and time (*VILT*).

The fundamental equations which Mr. Lowry gives for the MKS system at the end of his paper are erroneous. For instance, in common with all other systems, and by definition, $\mathbf{B} = \mu \mathbf{H}$. Again, as in all rationalized systems, $F = m^2/4\pi\mu r^2$.

For a more extended treatment of this subject the reader is referred to Professor Bradshaw's book.

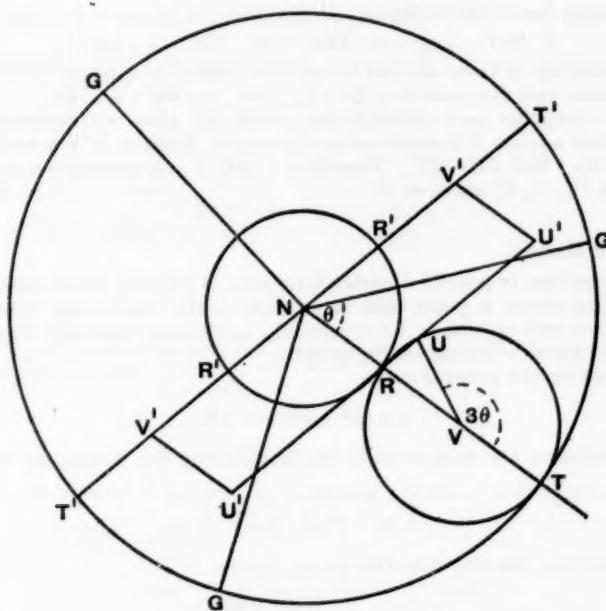
A. N. BLACK

REFERENCES

R. V. Lowry, "Units and Dimensions", *Mathematical Gazette* 38, (1954), 181-184.
 E. Bradshaw, *Electrical Units*, Chapman and Hall, 1952.

2634. On Note 2466 (Tangent-intercept property of the three cusped hypocycloid).

The following is a variation on Mr. Hope-Jones's proof, using multiple-valued angles. The phrase "the angle ABC " means "any turn from BA to BC , measured anti-clockwise".



Let the cusps of the hypocycloid be the three points G on a circle centre N , radius $3r$. A point U in the hypocycloid is constructed by drawing NV , of length $2r$, so that angle $VNG = \theta + n \cdot 120^\circ$, and VU , of length r , so that angle $T'VU = 3\theta + n \cdot 360^\circ$, where NV is produced to meet the circle at T . If a circle drawn with centre N and radius r meets NT at R , RU is a tangent to the hypocycloid (T being the instantaneous centre of the rolling circle). RU is produced both ways to the two points U' , where $RU' = 2r$. It is sufficient to prove that the points U' lie on the hypocycloid.

The parallelograms $NRU'V'$ are completed, and the lines NV' are produced to meet the original circle at T' and the circle whose centre is N and radius r at R' . Since $NV' = 2r$ and $V'U' = r$, it is sufficient to prove that angle $T'V'U' = 3 \times$ angle $T'NG$.

$$\text{Angle } T'V'U' = \text{angle } U'RV = l \cdot 360^\circ - (\frac{3}{2}\theta + m \cdot 180^\circ).$$

$$\text{Angle } T'NG = \text{angle } T'NT + \text{angle } TNG,$$

$$= l \cdot 360^\circ - (\frac{3}{2}\theta + m \cdot 180^\circ) + (\theta + n \cdot 120^\circ),$$

$$= l \cdot 360^\circ - \frac{1}{2}\theta - m \cdot 180^\circ + n \cdot 120^\circ.$$

$$3 \times \text{angle } T'NG = k \cdot 360^\circ - \frac{3}{2}\theta - m \cdot 180^\circ = \text{angle } T'V'U'.$$

Some other properties may also be proved in this manner : Since the tangent to the hypocycloid at U makes with NG the angles

$$l \cdot 360^\circ - \frac{1}{2}\theta - m \cdot 180^\circ + n \cdot 120^\circ,$$

it follows that the tangent at either point U' makes with NG the angles

$$l \cdot 360^\circ - \frac{1}{2}\theta' - m' \cdot 180^\circ + n \cdot 120^\circ,$$

where $\theta' = \frac{3}{2}\theta + m \cdot 180^\circ$, and the two positions of U' are distinguished by m being odd or even.

Substituting for θ' we obtain

$$l \cdot 360^\circ - \frac{1}{2}(\frac{2}{3}\theta + m \cdot 180^\circ) - m' \cdot 180^\circ + n \cdot 120^\circ;$$

and by changing m from an odd to an even value it appears that the angle between these two tangents is $\frac{1}{2}(2h+1) \cdot 180^\circ$, i.e. $90^\circ + h \cdot 180^\circ$.

As these tangents pass through the points R' , they will intersect on the inner circle at a point S diametrically opposite to R (since $R'N$ is half $U'R$ and parallel to it). But $SR = RT$. Therefore $U'SU'T$ is a rectangle and the three normals at U' , U , U' meet at T .

E. H. LLOYD.

2635. Cubic Graphs.

In the teaching of graphs of cubic functions in schools, the symmetry of all cubic graphs about a point mid-way between the stationary values of the function does not appear to be specifically mentioned although it is a useful property to know when sketching graphs.

For, consider the general cubic :

$$y = ax^3 + bx^2 + cx + d,$$

by differentiation the mid-point of the line joining the stationary values is :

$$\left(\frac{-b}{3a}, \frac{2b^3}{27a^2} - \frac{cb}{3a} + d \right),$$

and transferring the origin to this point gives :

$$y = ax \left(x^2 - \frac{(b^2 - 3ac)}{3a^2} \right)$$

which is symmetrical about the new origin. If $b^2 < 3ac$ the stationary points are "imaginary" and this is also the condition for not more than one real root of the equation $y = k$ to be possible ; but the property of symmetry remains.

A. HURRELL.

2636. A Propos de la Note No. 1542 (*The Mathematical Gazette*, vol. 25, No. 266, 1941, pp. 242-243).

1. E. H. N. (*Prof. E. H. Neville*) a montré que la fonction $y(x)$, définie par la relation

$$(1) \quad y^3 + 3xy + 2x^3 = 0,$$

satisfait l'équation différentielle

$$(2) \quad x^2(1+x^2) \frac{d^2y}{dx^2} - \frac{3}{2}x \frac{dy}{dx} + y = 0.$$

Nous indiquons ici un procédé qui conduit naturellement à la vérification du fait mentionné et donne la possibilité de trouver la solution générale de (2).

2. Au lieu de (1), on peut considérer

$$x = -\frac{3t}{t^2 + 2}, \quad y = -\frac{3t^2}{t^2 + 2}, \quad (t \text{ paramètre}),$$

d'où

$$\frac{dy}{dx} = \frac{t(t^2 - 4)}{2(t^2 - 1)}, \quad \frac{d^2y}{dx^2} = \frac{(t^2 + 2)^4}{12(t^2 - 1)^3}.$$

En portant ces expressions dans (2), on vérifie l'assertion, démontrée par E. H. N.

3. Si, dans (2), on fait le changement $x = -3t/(t^3 + 2)$, on obtient

$$\frac{d^2y}{dt^2} + f(t) \frac{dy}{dt} + g(t)y = 0,$$

où

$$f(t) = \frac{3t^2(t^3 - 4)}{(t^3 + 2)(t^3 - 1)} + \frac{3(t^3 + 2)^2(t^3 - 1)}{t[(t^3 + 2)^3 - 27t^3]},$$

$$g(t) = \frac{4(t^3 + 2)(t^3 - 1)^2}{t^2[(t^3 + 2)^3 - 27t^3]}.$$

Par suite, on obtient la solution générale de l'équation (2) sous une forme paramétrique que voici

$$x = -3t/(t^3 + 2),$$

$$y = C_1 y_1(t) + C_2 y_2(t),$$

avec C_1, C_2 constantes d'intégration, $y_1(t)$ et $y_2(t)$ étant définis par expressions :

$$y_1(t) = -3t^2/(t^3 + 2),$$

$$y_2(t) = y_1(t) \int \frac{dt}{y_1^2(t)} \exp(-\int f(t)dt).$$

DANICA PERČINKOVA-V'ČKOVA

2637. Un problème élémentaire sur la parabole.

Dans un système d'axes rectangulaires Ox, Oy , on donne les trois points : $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$, en supposant que :

$$x_1 \neq x_2 \neq x_3; \quad y_1 \neq y_2 \neq y_3.$$

La parabole passant par les points M_1, M_2, M_3 et dont l'axe de symétrie est parallèle à l'axe des x est définie par

$$(1) \quad x = \frac{(y - y_2)(y - y_3)}{(y_1 - y_2)(y_1 - y_3)} x_1 + \frac{(y - y_1)(y - y_3)}{(y_2 - y_1)(y_2 - y_3)} x_2 + \frac{(y - y_1)(y - y_2)}{(y_3 - y_1)(y_3 - y_2)} x_3.$$

La parabole passant par les points M_1, M_2, M_3 et dont l'axe de symétrie est parallèle à l'axe des y est définie par

$$(2) \quad y = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} y_3.$$

Ces deux paraboles se coupent en quatre points parmi lesquels les trois suivants sont connus sans aucun calcul : $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$. Les coordonnées du quatrième point se trouvent à partir des équations (1) et (2). En effet, si l'on élimine y des équations (1) et (2), on trouve

$$(3) \quad ax^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon = 0.$$

D'autre part, l'élimination de x des équations (1) et (2) fournit

$$(4) \quad ay^4 + by^3 + cy^2 + dy + e = 0.$$

Les coefficients $\alpha, \dots, \epsilon; a, \dots, e$ sont des fonctions simples rationnelles des $x_1, x_2, x_3, y_1, y_2, y_3$.

En utilisant les formules de Viète, les équations (3) et (4) fournissent :

$$(5) \quad x_4 = 2 \begin{vmatrix} 1 & 1 & 1 \\ x_1^2 & x_2^2 & x_3^2 \\ y_1 & y_2 & y_3 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} - (x_1 + x_2 + x_3),$$

$$(6) \quad y_4 = 2 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1^2 & y_2^2 & y_3^2 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} - (y_1 + y_2 + y_3).$$

Les deux dernières formules déterminent les coordonnées x_4 et y_4 du quatrième point M_4 .

Les faits précédents montrent que les paraboles (1) et (2) se coupent en quatre points M_1, M_2, M_3, M_4 , parmi lesquels les trois premiers sont donnés par avance, tandis que les coordonnées du quadratième sont déterminés par les formules (5) et (6) qui sont fort symétriques et jolies.

Beograd, Jugoslavija.

OLGA D. MITRINOVIC

2638. Problem arising from the Crap-game.

The rules of the Crap-Game are as follows. "It is played with two ordinary dice. The man in possession of them wins immediately if the total of the spots on his first throw is 7 or 11 and loses immediately if this total is 2, 3 or 12; in either case he continues to throw the dice. If the total is any one of the remaining six possible values he continues to roll the dice until he has either duplicated his own first throw, in which case he wins, or has thrown a total of 7, in which case he loses and is then required to surrender the dice to the other player."

It is reasonable to ask the question : how many times does a player expect to throw before handing the dice to the other player?

At any throw the probabilities of throwing the possible values are given by :

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Suppose that the initially expected number of throws is u .

If the first throw is 4 or 10 let the expected number of subsequent throws be denoted by v_1 .

Similarly if the first throw is 5 or 9 let the expected number of subsequent throws be v_2 and if the first throw is 6 or 8 let the expected number of subsequent throws be v_3 .

If the first throw is 2, 3, 7, 11 or 12 the game starts again at the second throw. In this case the expected number of subsequent throws is the same as the initially expected number and is therefore given by u .

To find the average expected number of throws subsequent to the first throw we must multiply the probability of the various possible first throws by the above expected numbers of subsequent throws and then add the products.

This gives for the average number of throws after the first

$$\frac{1}{6}v_1 + \frac{2}{6}v_2 + \frac{5}{6}v_3 + \frac{1}{6}u$$

But we must have that the initially expected number of throws = 1 + average number of throws after the first

i.e.

$$u = 1 + \frac{1}{6}v_1 + \frac{2}{6}v_2 + \frac{5}{6}v_3 + \frac{1}{6}u$$

In the same way we can consider the expected number of throws after the second throw. Suppose, first, that the first throw had been a 4. Then at the second throw there are three different possibilities :

- a throw of 4, with probability $\frac{1}{12}$, leads to a new start ;
- a throw of 7, with probability $\frac{1}{4}$, ends the sequence of throws ;
- all other throws, with probability $\frac{1}{2}$, leave the subsequently expected number of throws the same as before the second throw.

Again noting that the expected number of throws after the first throw is one greater than the average number after the first throw we have :

$$v_1 = 1 + \frac{1}{12}u + \frac{1}{2} \cdot 0 + \frac{1}{2}v_1$$

Similarly

$$v_2 = 1 + \frac{1}{6}u + \frac{1}{2} \cdot 0 + \frac{1}{2}v_2$$

and

$$v_3 = 1 + \frac{5}{6}u + \frac{1}{2} \cdot 0 + \frac{1}{2}v_3.$$

Solving the four simultaneous equations we find that u = initially expected number of throws = $8\frac{193}{196}$.

University College of the West Indies.

J. C. BARTON

2639. A note on ${}_aF_3$.

In Note 2499, Mr. Ross used partitions to prove the formula

$${}_3F_2 \left[\begin{matrix} 1, M, -M+m; \\ -M+1, M-m+1. \end{matrix} \right] = \frac{1}{2} \left\{ 1 + \frac{(2M-1)!}{(2m-1)!(2M-2m)!} \left[\frac{(M-m)!(m-1)!}{(M-1)!} \right]^2 \right\}, \quad \dots \dots \dots (1)$$

where M and m are positive integers and $M \geq m$. He also gave the sum of another ${}_3F_2$ which he showed could be deduced from Dixon's theorem. It is perhaps worth while noting that (1) can also be deduced from a known result.

We begin with the well-known formula*

$$\begin{aligned} {}_4F_3 & \left[\begin{matrix} a, 1 + \frac{1}{2}a, c, d, e; \\ \frac{1}{2}a, 1 + a - c, 1 + a - d, 1 + a - e \end{matrix} \right] \\ &= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)} \quad \dots \dots \dots (2) \end{aligned}$$

which reduces to Dixon's theorem when $e = \frac{1}{2}a$. Now let $a \rightarrow 0$ in (2) and we get

$$1 + 2 \sum_{r=1}^{\infty} \frac{(c)_r (d)_r (e)_r}{(1-c)_r (1-d)_r (1-e)_r} = \frac{\Gamma(1-c)\Gamma(1-d)\Gamma(1-e)\Gamma(1-c-d-e)}{\Gamma(1-d-e)\Gamma(1-c-e)\Gamma(1-c-d)}. \quad \dots \dots \dots (3)$$

This is equivalent to a formula† given by Dougall nearly fifty years ago and described by Hardy as a "particularly elegant formula".

If $e = -N$, where N is a positive integer, (3) becomes

$$1 + 2 \sum_{r=1}^{N} \frac{(c)_r (d)_r (-N)_r}{(1-c)_r (1-d)_r (1+N)_r} = \frac{N! (1-c-d)_N}{(1-c)_N (1-d)_N} \quad \dots \dots \dots (4)$$

and, with $d = \frac{1}{2}$, this reduces to

* See W. N. Bailey, *Generalised hypergeometric series* (Cambridge, 1935), § 4.4.

† See loc. cit., p. 96, ex. 1 (viii), where references are given.

$$1 + 2 \sum_1^N \frac{(c)_r (-N)_r}{(1-c)_r (1+N)_r} = \frac{N! (\frac{1}{2} - c)_N}{(\frac{1}{2})_N (1-c)_N}. \dots \quad (5)$$

The formula (5) from which (1) has been deduced is also a limiting case of the formula

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n) \Gamma(d+n)} = \frac{\pi^2 \Gamma(c+d-a-b-1)}{\sin \pi a \sin \pi b \Gamma(c-a) \Gamma(d-a) \Gamma(c-b) \Gamma(d-b)}$$

given by Dougall in 1907 *, with b, d replaced by $-N, 1+N$ and then a, c replaced by $c, 1-c$.

Now take $c=M$, a positive integer $> N$. Then (5) becomes

$$1 + 2 \sum_1^N \frac{(M)_r (-N)_r}{(1-M)_r (1+N)_r} = \frac{(2M-1)!}{(2M-2N-1)! 2N!} \left[\frac{N! (M-N-1)!}{(M-1)!} \right]^2$$

and this gives (1) on writing $N=M-m$.

It will be noticed that both the sums given by Mr. Ross are limiting cases of (2).

W. N. BAILEY

2640. Notes on Conics. No. 20. The distance-quadratic in pure geometry.

Let a line through a point O cut the directrix in R , and let U be the projection of the focus S on this line. Then if P is any point of the line,

$$SP^2 = OP^2 - 2OU \cdot OP + OS^2, \quad RP = OP - OR.$$

It follows that if e_θ is the oblique eccentricity associated with the slope of the line OR , then P is on the conic if and only if

$$(1 - e_\theta^2)OP^2 - 2(OU - e_\theta^2 OR)OP + (OS^2 - e_\theta^2 OR^2) = 0.$$

Hence if the line cuts the conic in two points P, Q , the midpoint V of PQ is determined by the condition

$$VU = e_\theta^2 VR,$$

and the product $OP \cdot OQ$ is given by

$$OP \cdot OQ = \Gamma_0 / (1 - e_\theta^2),$$

where, since $e_\theta | OR |$ is the radius of the eccentric circle of O , the numerator Γ_0 is the power of S for that circle. E. H. N.

2641. A conditional algebraic identity.†

Surely nothing but Cayley's ineptitude in the matter could have persuaded anyone that the Tripos problem

Prove that, if $a+b+c=0$ and $x+y+z=0$, then

$$4(ax+by+cx)^3 - 3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) \\ - 2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y) = 54abcxyz$$

is anything but the simple exercise in elementary algebra which its place in the examination declared it to be. Writing

* Proc. Edinburgh Math. Soc. 25 (1907), 114-132.

† See M. O. XXXIX, pp. 280-286 for a discussion and further references. Cayley's solution is reprinted in his *Papers* as No. 758, occupying XI, 265-267.

$$\begin{aligned}\phi &= 4(ax+by+cz)^3 - 3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) \\ \psi &= (b-c)(c-a)(a-b)(y-z)(z-x)(x-y) + 27abcxyz\end{aligned}$$

we are to establish the conditional identity $\phi = 2\psi$, knowing that $ax+by+cz$, which we will denote by D , is a factor of ϕ , and that, since in ψ the parameters are not attached individually to the variables, if $ax+by+cz$ is a factor of ψ , so also are $bx+cy+az$ and $cx+ay+bz$. It follows that our problem is to express ϕ and ψ as constant multiples of the product of three known factors.

In ϕ the key is held out, if Professor Watson will allow me to say so, by the product $(a^2+b^2+c^2)(x^2+y^2+z^2)$, which clamours to be replaced by

$$(ax+by+cz)^2 + (cy-bz)^2 + (az-cx)^2 + (bx-ay)^2.$$

What both Professor Watson and Dr. Lawrence have overlooked is that, under the conditions, the last three terms in this sum are equal. With α, β, γ for $b-c, c-a, a-b$ we have

$$D = \beta z - \gamma y = \gamma x - \alpha z = \alpha y - \beta x,$$

and since $\beta - \gamma = -3a$, — and how often this is the useful deduction from $a+b+c=0$, — we have also, if $E = \alpha x + \beta y + \gamma z$, then

$$\frac{1}{2}E = cy - bz = az - cx = bx - ay.$$

Hence

$$\begin{aligned}\phi &= D(D^2 - E^2) \\ &= 4(ax+by+cz)(bx+cy+az)(cx+ay+bz).\end{aligned}$$

On the other hand,

$$\psi = \alpha\beta\gamma(y-z)(z-x)(x-y) - xyz(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

If in ψ we replace x by $-y-z$, then ψ becomes a polynomial in the independent variables y, z , and the elementary criterion for a factor applies: $qy-pz$ is a factor if $\psi=0$ when $y=p$ and $z=q$. It follows that $qy-pz$ is a factor of ψ in its original form when $x+y+z=0$ if $\psi=0$ when $x=-p-q, y=p, z=q$. Now $\psi=0$ for $x=\alpha, y=\beta, z=\gamma$, for $x=\beta, y=\gamma, z=\alpha$, and for $x=\gamma, y=\alpha, z=\beta$; hence ψ is a multiple of the product $(yy-\beta z)(\alpha y-\gamma z)(\beta y-\alpha z)$. To find the constant factor, we pick out the term in y^3 in ψ after substitution for x ; that is, we write $x=-y, z=0$ in ψ , obtaining $-2\alpha\beta\gamma y^3$. Thus the factor required is -2 , and we have

$$\begin{aligned}\psi &= 2(\beta z - \gamma y)(\gamma z - \alpha y)(\alpha z - \beta y) \\ &= 2(ax+by+cz)(bx+cy+az)(cx+ay+bz)\end{aligned}$$

completing the proof.

An unargumentative factorisation of ψ is worth recording. We have

$$\begin{aligned}\psi &= -\alpha\beta\gamma\{yz(y-z) + zx(z-x) + xy(x-y)\} \\ &\quad + xyz\{\beta\gamma(\beta-\gamma) + \gamma\alpha(\gamma-\alpha) + \alpha\beta(\alpha-\beta)\},\end{aligned}$$

and therefore, since $\alpha(y-z) - (\beta-\gamma)x = 2D$,

$$\begin{aligned}\psi &= 2D\{\beta\gamma yz + \gamma\alpha zx + \alpha\beta xy\} \\ &= 2D\{\alpha\beta y^3 - (\beta\gamma + \alpha^2)yz + \alpha\gamma z^3\} \\ &= 2D(\gamma z - \alpha y)(\alpha z - \beta y).\end{aligned}$$

E. H. N.

Editorial note. Very simple solutions of this problem were received also from Mr. B. Spain and Mr. W. J. Price.

2642. Unitary Construction of Certain Polyhedra.

The purpose of this note is to describe a method by which several Archimedean polyhedra can be built up from a number of identical blocks of quite simple construction. In some cases interesting combinations of polyhedra are obtained. The method is very suitable for use as a class experiment, provided sufficient accuracy can be achieved.

Most people are familiar with the construction of the cube from six square pyramids with a common vertex at the centre. Six such pyramids placed outwards on a cube form a rhombic dodecahedron. Obviously any kind of Archimedean polyhedron can be constructed in a similar manner, using as many types of pyramids as there are types of faces, but there is no great advantage in this. In the method now to be described, the polyhedra are hollow, and in many cases the interiors can be seen since the solids are pierced by holes.

The simplest case is the truncated tetrahedron ($3 \cdot 6^2$). Construct four units A as in fig. 1. Each unit has for base a regular hexagon. Alternate sloping sides are right-angled isosceles triangles and rectangles with sides in the ratio $1 : \frac{1}{2}\sqrt{2}$ ($2 \sin \pi/4 : 1$). An equilateral triangle closes the top. The net is obvious. If four such boxes are stuck together by their rectangular faces they will form a truncated tetrahedron with a tetrahedral hollow in the interior. The tetrahedron cannot of course be seen unless one box is omitted. If eight are stuck together by their triangular faces, however, a truncated octahedron results with cuboidal holes where its square faces should be, leading to a cuboctahedral interior.

FIG. 1.

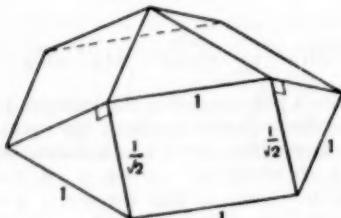


FIG. 2.

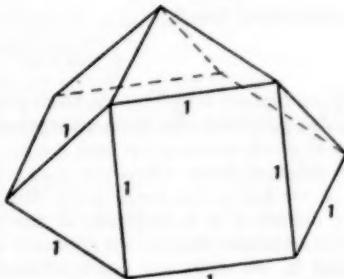


FIG. 3.

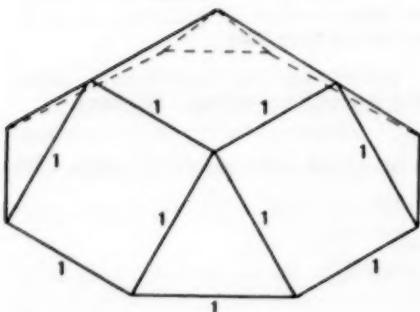
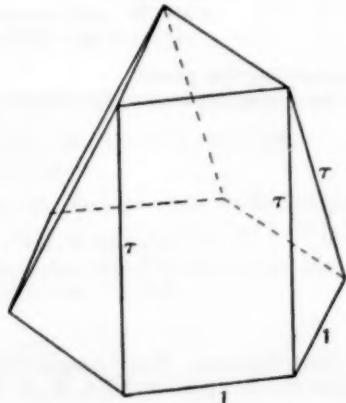


FIG. 4.



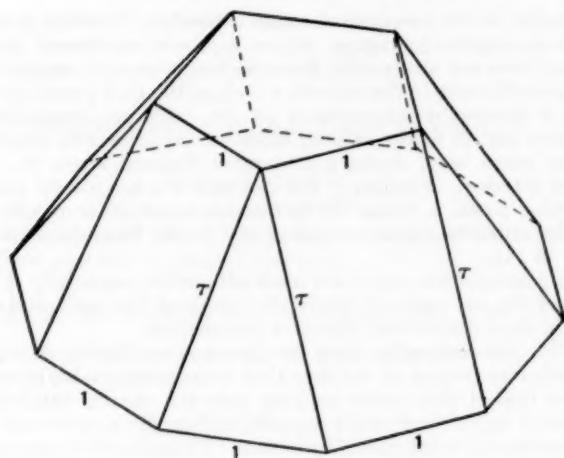


FIG. 5.

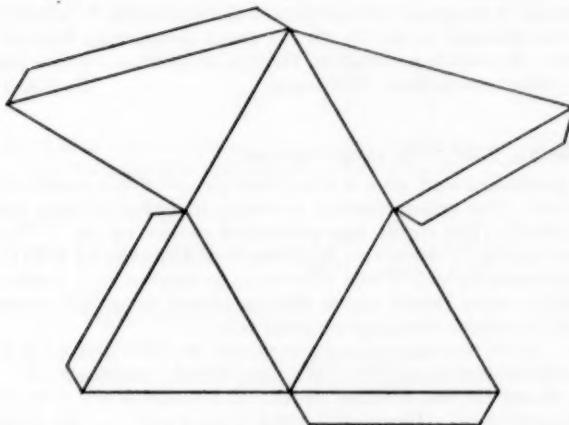


FIG. 6.

For the solids with octahedral symmetry there are two kinds of unit. The *B*-units (fig. 2) have a resemblance to the *A*'s, but the sloping rectangles have sides in the ratio $2 \sin \pi/6 : 1$, and are therefore squares, so that the sloping triangles are equilateral. The units are halves of cuboctahedra. Eight *B*'s, joined by their square faces, make a truncated octahedron ($4 \cdot 6^2$) with its square faces missing, revealing the vertices of an octahedral interior. The *C*-units (fig. 3) are caps of Small Rhombicuboctahedra ($3 \cdot 4^3$). They have regular octagonal bases, square tops, and their sloping faces are again squares and equilateral triangles. Six *C*'s, stuck together by their square faces, make a truncated cube ($3 \cdot 8^2$) with a cubical interior. If now the eight *B*'s and six *C*'s are stuck together alternately by their triangular faces, a beautiful solid is produced, consisting of a Great Rhombicuboctahedron ($4 \cdot 6 \cdot 8$) with its square faces removed, revealing cubical holes leading to a Small Rhombicuboctahedron ($3 \cdot 4^3$).

The treatment of the icosahedral solids is similar. Twenty D -units (fig. 4) whose bases are regular hexagons, whose tops are equilateral triangles, and whose sloping faces are alternately isosceles triangles with angles $36^\circ, 72^\circ, 72^\circ$ and rectangles with sides in the ratio $1 : \tau(2 \sin \pi/10 : 1)$, if joined by rectangular faces, form a truncated icosahedron ($5 . 6^2$) with an icosahedral interior. Twelve E -units (fig. 5) whose sloping faces are congruent to those of the D 's, joined in the same way, make a truncated dodecahedron ($3 . 10^2$) with a dodecahedral interior. Finally, if the D 's and E 's are joined alternately by their triangular faces, a Great Rhombicosidodecahedron results ($4 . 6 . 10$), penetrated by cuboidal holes, revealing the Small Rhombicosidodecahedron inside ($3 . 4 . 5 . 4$).

These two combination solids are most attractive, especially if the B 's and C 's, or D 's and E 's, are made of differently coloured thin card, and when a light is hung inside they make most effective decorations.

Incidentally, the realization that the boxes B are halves of cuboctahedra, combined with recognition of the fact that truncated octahedra can fill space, enables us to dissect this latter packing into the central octahedra and the boxes B , then to unite the B -units in pairs, and finally to arrive at the packing by cuboctahedra and octahedra alternately. This process is somewhat similar to that which derives the rhombic dodecahedral packing from the cubic via the six square pyramids.

The dissection of a regular tetrahedron into four units F , whose net is given in fig. 6, communicated to me by Mr. Dorman Luke, may be new to readers of the *Gazette*. It makes a pleasing, though somewhat simple puzzle to put together the tetrahedron from four units.

H. MARTYN CUNDY

2643. Note on No. 2505. On magic squares.

This will generalize and give a simplified proof of the result of A. D. and K. H. V. Booth. The generalization is clearly included in their ideas, but not the simpler proof. The result was published earlier by me ("The row-sums of the inverse matrix", *American Mathematical Monthly*, 58 (1951), 614-615); and some questions (which I was later able to answer in a forthcoming note in the *Monthly*) were raised about the extension to infinite matrices. The present proof is simpler than my original one.

Theorem : If the row-sums of a finite matrix are all c and $c \neq 0$, then the row-sums of the inverse matrix are $1/c$. We may clearly assume $c = 1$.

Letting $\mathbf{1}$ stand for the column vector of 1's and $B = A^{-1}$ we have $A\mathbf{1} = \mathbf{1}$ (this is the hypothesis). Hence $B\mathbf{1} = BA\mathbf{1} = I\mathbf{1} = \mathbf{1}$. Q. E. D.

Lehigh University, Pennsylvania.

ALBERT WILANSKY

2644. Limits of sequences and a theorem of L'Hospital.

In this note we prove some well-known elementary limit properties of sequences of real numbers using a somewhat modified theorem of l'Hospital. We use the following :

Lemma.

Let $h(x)$ be a function, which is continuous and has a right-hand derivative $h'(x)$ on $p \leq x \leq q$. If $h(p) = h(q) = 0$ and $h(x) \neq 0$ then we have $\sup h'(x) > 0$ and $\inf h'(x) < 0$.

Proof.

Let $\inf h(x) = h(a) < 0$, then $h'(a) \geq 0$, thus $\sup h'(x) \geq 0$. If $\inf h(x) = 0$, then $h'(p) \geq 0$, thus $\sup h'(x) \geq 0$. Using $-h(x)$ we can deduce without any difficulty that $\inf h'(x) \leq 0$.

If there is a c with $h(c) > 0$, we define

$$k(x) = h(x) - \frac{x-p}{c-p} h(c)$$

then $k(c) = k(p) = 0$, thus $\sup k'(x) \geq 0$ and $\sup h'(x) \geq h(c) > 0$.

Using

$$l(x) = h(x) - \frac{q-x}{q-c} h(c)$$

we find $\inf h'(x) \leq -h(c) < 0$. If there is no c with $h(c) > 0$, there is a c with $h(c) < 0$ and we proceed by similar methods.

Theorem.

Let, for $x > A$, the function $g(x)$ be differentiable, $g'(x) > 0$, $\lim_{x \rightarrow \infty} g(x) = \infty$; let $f(x)$ be continuous and have a right-hand derivative $f'(x)$.

Then for $x \rightarrow \infty$ we have

$$\underline{\lim} \frac{f'(x)}{g'(x)} \leq \underline{\lim} \frac{f(x)}{g(x)} \leq \overline{\lim} \frac{f(x)}{g(x)} \leq \overline{\lim} \frac{f'(x)}{g'(x)}$$

Proof.

We introduce, for $A < p \leq x \leq q$,

$$h(x) = \begin{vmatrix} f(x) & g(x) & 1 \\ f(p) & g(p) & 1 \\ f(q) & g(q) & 1 \end{vmatrix},$$

then $h(x) \equiv 0$ if and only if $f(x) = \lambda g(x) + \mu$ for every $x > A$. If we delete this trivial case, $h(x)$ satisfies the conditions of the lemma.

Thus

$$\sup h'(x) > 0,$$

$$\sup \left\{ [g(q) - g(p)] \cdot g'(x) \cdot \left[\frac{f'(x)}{g'(x)} - \frac{f(q) - f(p)}{g(q) - g(p)} \right] \right\} > 0,$$

and since

$$[g(q) - g(p)] \cdot g'(x) > 0$$

we have

$$\sup \frac{f'(x)}{g'(x)} > \frac{f(q) - f(p)}{g(q) - g(p)}, \quad (p \leq x \leq q)$$

Now we divide the right-hand side by $g(q)$ and let $q \rightarrow \infty$, then

$$\overline{\lim}_{q \rightarrow \infty} \frac{f(q)}{g(q)} \leq \sup \frac{f'(x)}{g'(x)} \quad (x \geq p)$$

Let now $p \rightarrow \infty$, then

$$\overline{\lim} \frac{f(x)}{g(x)} \leq \overline{\lim} \frac{f'(x)}{g'(x)}.$$

The further proof of the theorem is now obvious.

We give some applications. If a_1, a_2, \dots is a sequence of real numbers, we define $f_1(n) = \sum_{k=1}^n a_k$ if n is a positive integer. For other positive values of x we define $f_1(x)$ by linear interpolation. If a_1, a_2, \dots is a sequence of positive numbers, we define $f_2(x) = \log a_n$ if n is a positive integer and we define $f_2(x)$ for other positive values of x again by linear interpolation.

Let $g_1(x) = x$ and $g_2(x) = \log x$. The functions f_1, f_2, g_1, g_2 satisfy the conditions of our theorem. After some manipulations we obtain the following inequalities :

$$(1) \quad \underline{\lim} a_n \leq \underline{\lim} \frac{1}{n} \sum_{k=1}^n a_k \leq \overline{\lim} \frac{1}{n} \sum_{k=1}^n a_k \leq \overline{\lim} a_n \dots (f_1, g_1)$$

$$(2) \quad \underline{\lim} n a_n \leq \underline{\lim} \frac{1}{\log n} \sum_{k=1}^n a_k \leq \overline{\lim} \frac{1}{\log n} \sum_{k=1}^n a_k \leq \overline{\lim} n a_n \dots (f_1, g_2)$$

$$(3) \quad \underline{\lim} \frac{a_{n+1}}{a_n} \leq \underline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \frac{a_{n+1}}{a_n}, \dots (f_2, g_1),$$

$$(4) \quad \underline{\lim} n \log \frac{a_{n+1}}{a_n} \leq \underline{\lim} \frac{\log a_{n+1}}{\log n} \leq \overline{\lim} \frac{\log a_n}{\log n} \leq \overline{\lim} n \log \frac{a_{n+1}}{a_n} \dots (f_2, g_2),$$

(the pair of functions in brackets denoting the choice of f and g in that line).

The last one can be used for proving the convergence criterium : a series $a_1 + a_2 + a_3 + \dots$ is convergent if $\lim_{n \rightarrow \infty} n \log \frac{a_{n+1}}{a_n} < -1$.

Rijks-Universiteit, Utrecht.

F. VAN DER BLIJ

2645. On Note 2548.

An alternative test for divisibility by 19 has been suggested by my pupil P. J. Roberts.

Since $a = 20q + r = 19q + q + r$, $a \equiv 0 \pmod{19}$ if $q + r \equiv 0 \pmod{19}$ where q, r are the quotient and remainder respectively when a is divided by 20. This process can be repeated until it is evident whether or not the sum of the quotient and remainder is divisible by 19. For the example given in Note 2548, the steps are

$$\begin{array}{r} 3086379 \\ 154337 \\ 7733 \\ 399 \\ 38 \\ \hline 3086379 \equiv 0 \pmod{19}. \end{array}$$

This method can be readily adapted to test divisibility by any number of the form $10n - 1$.

G. MATTHEWS

2646. On note 2571.

In my note 2385, on which Miss C. Hamill comments, I should have stated that the coordinates were Cartesian $x : z$ and $y : z$. However, since all my note except the first result is valid for general homogeneous coordinates, Miss Hamill's emendation is certainly an improvement.

Perhaps the form which I quoted for the equation of the chord $P_1 P_2$, namely

$$S_1 + S_2 - S_{12} = 0$$

is not so well known as it deserves to be. The proof is immediate, since this equation in Cartesian coordinates is of the first degree and is obviously satisfied by the coordinates of P_1 and of P_2 . This equation gives at once what must surely be the shortest proof for the equation of the tangent at P_1 .

F. M. GOLDNER

2647. A mathematical tile.

There appeared in the *Daily Telegraph* for June 30, 1955, in the column "London Day by Day", a most interesting example of a curved tile and a repeating pattern made from it. These were displayed by the Italian Institute at their exhibition of Italian Industrial Design. The interesting thing about the drawing is that the tile depicted had evidently been slightly altered in the course of reproduction, and, as printed, could not have formed a unit in a repeating pattern. Indeed, close inspection of the pattern reproduced alongside revealed that the tiles were not exactly of the shape shown.

The true shape of the tile is evidently based on a parallelogram unit cell, consisting of two adjoining 60° rhombs. (Fig. 1). Circular arcs are described on the sides of these rhombs, as the diagram indicates. "Peterborough" commented : "The curves are calculated with a skill that Leonardo, that forerunner of all mathematically-minded designers, would have envied", but this seems a slight over-statement!

Part of the pattern reproduced in the *Daily Telegraph* is shown in Fig. 2. Such mass-produced tiles are evidently available in Italy. One would like to know if any English firm is sufficiently go-ahead to produce something similar ; such designs are of course endless.

The diagrams are reproduced from blocks lent by *Art Club* who published a short article on the tiles in their third issue. From them I learn the additional information that the tiles were designed by Alberto Scarzella and Marco Zanuso, whose initials give the tile its name of S.Z.I. H. M. C.

FIG. 1.

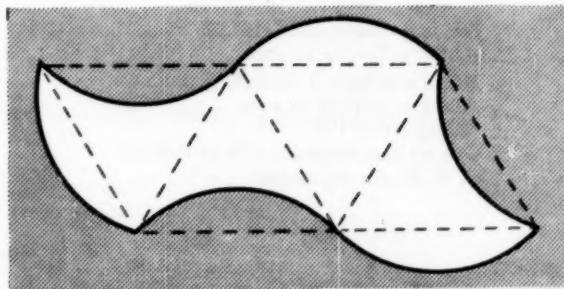
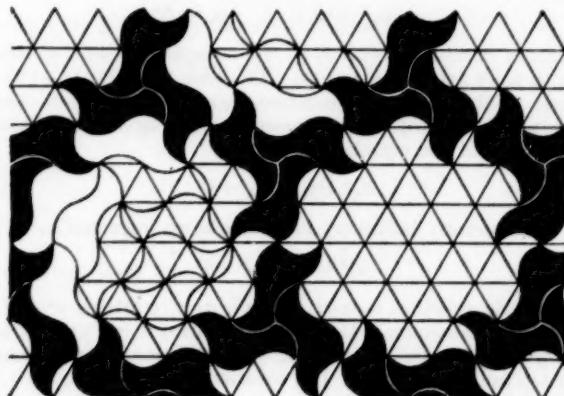


FIG. 2.



2648. Collinearity of the diagonal middle points of the complete quadrilateral.

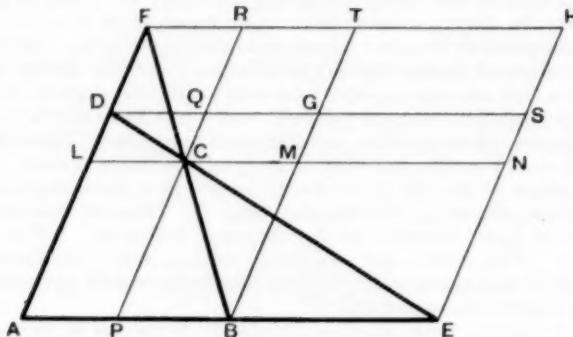


FIG. 1.

To prove that the middle points of the diagonals of a complete quadrilateral $ABCDEF$ are collinear.

In the figure the lines $LCMN$, $DQGS$ and $FRTH$ are all parallel to AB , while $PCQR$, BMG and $ENSH$ are parallel to AD .

Proof. The mid-points of the diagonals of the quadrilateral are the intersections of the diagonals of the parallelograms AC , AG , AH and are collinear if C , G , H are collinear.

Since DE is a diagonal of $ADSE$
 $\therefore ALCP = CNSQ$ in area

Since BF is a diagonal of $ABTF$
 $\therefore ALCP = CMTR$ in area
 $\therefore CNSQ = CMTR$
 $\therefore G$ is on the diagonal CH of $CNHR$
 $\therefore C, G, H$ are collinear.

B. M. PECK

2649. On note 2492.

Dr. Booth's solution of the equation

$$a_1^2 + a_2^2 + \dots + a_n^2 = b^2$$

is not, in the form in which he gives it, the complete primitive solution unless one allows λ to take fractional values. The case $1^2 + 1^2 + 1^2 + 1^2 = 2^2$ is otherwise excluded. Only in the Pythagorean case is the solution complete.

I came across this incompleteness recently when investigating the existence of integer solutions of the equation

$$a_1^4 + a_2^4 + a_3^4 = b^4,$$

an anvil on which many hammers have made no impression. The equation

$$a_1^4 + a_2^4 + a_3^4 = b^2 \quad \dots \dots \dots \quad (1)$$

has the parametric solution $a_1 = yz$, $a_2 = zx$, $a_3 = xy$ where $x^2 + y^2 = z^2$, but in this case b cannot be square. If x, y, z is a primitive Pythagorean triangle, the solution y^2z^2, z^2x^2, x^2y^2 of $a_1^4 + a_2^4 + a_3^4 = b^4$ can be represented by Dr. Booth's solution only with $\lambda = \frac{1}{2}$. The solution of (1) given by $x = 3, y = 4, z = 5$ was known to Diophantus (*Arithmetica*, v. 29). JOHN LEECH

2650. A generalisation of the formula for multidimensional Pythagorean numbers.

Booth's generalisation of the formula for Pythagorean numbers (Note 2492) is a special case of a less familiar and equally simple solution of the following problem :

Given integers m_1, m_2, \dots, m_n , to determine integers $(A, B_1, B_2, \dots, B_n, C)$ such that

$$A^2 + m_1 B_1^2 + m_2 B_2^2 + \dots + m_n B_n^2 = C^2.$$

Solution.

$$A = \lambda(\sum m_r b_r^2 - a^2), \quad B_r = 2\lambda ab_r, \quad C = \lambda(\sum m_r b_r^2 + a^2) \quad \dots \dots \dots (1)$$

where the sums run from 1 to n .

Booth's formula is obtained from the above when we put

$$m_1 = m_2 = \dots = m_n = 1.$$

It is noteworthy that the above formula remains valid when some of the m 's are negative (the formula does not in fact give all solutions).

Applications. In many cases the above formula readily yields solutions of Diophantine equations which are usually solved by more elaborate methods. The following examples are offered as illustrations.

Problem I. Solve the Diophantine equation

$$x^2 + y^2 = z^2 + u^2 \quad \dots \dots \dots (2)$$

Solution. Write

$$x^2 + y^2 - z^2 = u^2$$

Here $m_1 = 1, m_2 = -1$. Hence by substitution in (1)

$$\begin{aligned} x &= \lambda(a^2 - b^2 + c^2) \\ y &= 2\lambda ab \\ z &= 2\lambda ac \\ u &= \lambda(a^2 + b^2 - c^2). \end{aligned}$$

For $\lambda = 1$ we may write the solution in the form

$$(a^2 - b^2 + c^2)^2 + (2ab)^2 = (a^2 + b^2 - c^2)^2 + (2ab)^2$$

A comparison with the well known solution

$$(pq - rs)^2 + (pr + qs)^2 = (pq + rs)^2 + (pr - qs)^2$$

yields a number of interesting classroom exercises.

Problem II. Find a number of the form $m^2 + n^2$ which is also a sum of 3 squares.

Solution.

$$\begin{aligned} x^2 + y^2 + z^2 &= u^2 + v^2 \\ x^2 + y^2 + z^2 - u^2 &= v^2 \end{aligned}$$

Hence

$$\begin{aligned} x &= a^2 - b^2 - c^2 + d^2, \quad y = 2ab, \quad z = 2ac \\ u &= 2ad, \quad v = a^2 + b^2 + c^2 - d^2. \end{aligned}$$

2651. A trivial inequality.

The inequality is $4 \neq 2$. This inequality is unconnected with my work on inequalities which the *Gazette* has recently published but it was suggested by the paper on pp. 203-206 of No. 329 of the *Gazette*; the paper contains the solecism "salti" as the plural of *saltus passim*. The noun *saltus* belongs to what was described in my young days as the fourth declension (not the second), and so the plural is *saltus*. The metamorphosis is possibly to be attributed to the modern habit of travelling in omnibi. G. N. WATSON

2652. Schur's inequality and Watson's identities.*

This note deals with the symmetric function

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \dots \quad (1)$$

of three variables. Watson's identities and some others are established straightforwardly, and it is proved that the function is not expressible as a sum of non-negative products. Since the more general function with an arbitrary index μ is not introduced, the function written in full in (1) will be denoted by $\phi(x, y, z)$, or simply by ϕ , instead of by $f(x, y, z; 1)$.

Identically,

$$y(y-z)(y-x) + z(z-x)(z-y) = (y-z)^2(y+z-x), \dots \quad (2)$$

and therefore

$$\Sigma(y-z)^2(y+z-x) = 2\Sigma x(x-y)(x-z),$$

that is,

$$2\phi = \Sigma(y+z-x)(y-z)^2. \quad (3)$$

Arranged as a polynomial in x ,

$$\phi = x^3 - x^2(y+z) - x(y^2 + z^2 - 3yz) + (y+z)(y-z)^2.$$

In regrouping the terms, perhaps our first impulse is to subtract from 4ϕ the terms which compose $x(2x-y-z)^2$, but with $z=0$ we have the simple identity

$$\phi(x, y, 0) = (x^2 - y^2)(x-y) = x(x-y)^2 + y(x-y)^2,$$

and this suggests that we take out at once the sum

$$x(x-y-z)^2 + y(x-y)^2 + z(x-z)^2;$$

we find then

$$\phi = x(x-y-z)^2 + y(x-y)^2 + z(x-z)^2 - yz(y+z-x), \dots \quad (4)$$

an identity typical of a set of three.

Schur's inequality, $\phi \geq 0$ for non-negative x, y, z , follows immediately from (3) and (4), for if the three numbers $y+z-x, z+x-y, x+y-z$ are all non-negative, ϕ is given in (3) as the sum of three non-negative products, while if one of these three numbers is negative, ϕ is given as the sum of four non-negative products by the corresponding identity of the type of (4). Logically this proof seems symmetrical, in a sense in which a proof which begins "Suppose $x \leq y \leq z$ " does not, but there is no genuine distinction; from (1) and (2) we have

* *Math. Gazette*, Vol. XXXVII, p. 245; Vol. XXXIX, p. 207.

$$\phi = x(y-x)(z-x) + (y-z)^2(y+z-x), \dots \quad (5)$$

and instead of saying "Suppose $x \leq y \leq z$ " we have only to say that of the three identities typified by (5), there is certainly one which expresses ϕ as the sum of two non-negative products.

From (3), since

$$\begin{aligned} (x+y+z)(y+z-x) &= (y+z-x)^2 + 2x(y+z-x) \\ &= (y+z-x)^2 + 2yz - 2(x-y)(x-z), \end{aligned}$$

we have

$$2(x+y+z)\phi = \Sigma\{(y+z-x)^2 + 2yz\}(y-z)^2 - 2\Sigma(x-y)(x-z)(y-z)^2,$$

and since

$$\Sigma(x-y)(x-z)(y-z)^2 = -(y-z)(z-x)(x-y)\Sigma(y-z) = 0,$$

this becomes

$$2(x+y+z)\phi = \Sigma(y+z-x)^2(y-z)^2 + 2\Sigma yz(y-z)^2, \quad \dots \quad (6)$$

Watson's symmetrical identity of the fourth degree. The unsymmetrical identity from which Watson derives (6) presents itself if we combine the first and last products on the right of (4), for

$$x\{x(x-y-z) + yz\} = \phi + (x-y-z)(y-z)^2$$

from (5), and therefore

$$x\phi = (x-y-z)\phi + (x-y-z)^2(y-z)^2 + xy(x-y)^2 + xz(x-z)^2,$$

that is,

$$(y+z)\phi = (x-y-z)^2(y-z)^2 + xy(x-y)^2 + xz(x-z)^2 \quad \dots \quad (7)$$

as required.

If the non-negative numbers y, z are written as η^2, ζ^2 , the sum $y(x-y)^2 + z(x-z)^2$ is expressible as $u^2 + v^2$, where $u = \eta(x-y)$, $v = \zeta(x-z)$, and the elementary identity

$$(\eta^2 + \zeta^2)(u^2 + v^2) = (\eta u - \zeta v)^2 + (\zeta u + \eta v)^2$$

becomes

$$(y+z)(y(x-y)^2 + z(x-z)^2) = (x-y-z)^2(y-z)^2 + yz(2x-y-z)^2 \quad \dots \quad (8)$$

whence from (7) we have Watson's unsymmetrical identity of the fifth degree, namely

$$(y+z)^2\phi = (x+y+z)(x-y-z)^2(y-z)^2 + xyz(2x-y-z)^2. \quad \dots \quad (9)$$

Admittedly the identity (8) is not obvious if only the right-hand side is given, and Watson's process of division by $\eta + i\zeta$ was devised to secure the quotient by $y+z$ in the desired form of the sum of non-negative parts. We can however achieve this result with less labour and no less assurance. If, knowing the identity (9), we write

$$(x-y-z)^2(y-z)^2 + yz(2x-y-z)^2 = (y+z)\phi,$$

then ϕ is a non-negative cubic such that

$$(y+z)\phi = (x-y-z)^2(y-z)^2 + x\psi, \quad \dots \quad (10)$$

and since $(y+z)\phi$ is zero if two of the three variables are equal and the third is either zero or equal to these two, the same is true of the component $x\phi$. Hence $\psi=0$ if $z=0$ and $x=y$, if $y=0$ and $x=z$, and if $x=y=z$, and if ψ is dissected into any number of non-negative parts, the same is true of each part. There is no term of degree 3 in x in ψ ; the terms of degree 2 are x^2y and x^2z , and these can come only from products yu^2 and zv^2 , where u and v are linear. From the conditions just set down, u is a multiple of $x-y$ and v is a multiple of $x-z$. Hence, if ψ can be dissected, the sum

$$y(x-y)^2 + z(x-z)^2$$

must form part of the dissected expression. If this sum is not the whole of ψ , what is left is of the form $x\chi_2 + \chi_3$ where χ_2, χ_3 are polynomials in y and z of degrees 2 and 3; each of these vanishes if $y=0$, if $z=0$, and if $y=z$, and therefore, being non-negative for positive values of y and z , is a multiple of $yz(y-z)^2$, which is impossible. That is,

$$\psi = y(x-y)^2 + z(x-z)^2,$$

and (7) is recovered from (10).

If the equation whose roots are x, y, z is

$$F(t) = t^3 - pt^2 + qt - r = 0,$$

then

$$(x-y)(x-z) = F'(x) = 3x^2 - 2px + q,$$

and if s_1, s_2, s_3 are the sums $\Sigma x, \Sigma x^2, \Sigma x^3$ we have

$$\phi = 3s_2 - 2ps_1 + qs_3, \quad \dots \quad (11)$$

and since $p=s_1$ and $2q=s_1^2-s_2$,

$$2\phi = s_1^3 - 5s_1s_2 + 6s_3. \quad \dots \quad (12)$$

Also, since $(z-x)-(x-y)=p-3x$, the elementary identity

$$bc(b-c) + ca(c-a) + ab(a-b) = -(b-c)(c-a)(a-b)$$

implies

$$\Sigma(p-3x)(x-y)(x-z) = (p-3x)(p-3y)(p-3z),$$

and therefore

$$3\phi = p(3s_2 - 2ps_1 + 3q) - (p^3 - 3p^2 + 9pq - 27r),$$

that is,

$$\phi = p(s_2 - 2q) + 9r;$$

this identity, which comes immediately also from (11) by means of the familiar identity $s_3 - 3r = p(s_2 - q)$, gives

$$\phi = p^3 - 4pq + 9r. \quad \dots \quad (13)$$

The expressions for ϕ in (11), (12) and (13) facilitate the verification of identities. Consider for example (6) above. The equation whose roots are $p-2x, p-2y, p-2z$ is

$$(t-p)^3 + 2p(t-p)^2 + 4q(t-p) + 8r = 0,$$

that is,

$$t^3 - pt^2 - (p^2 - 4q)t + (p^3 - 4pq + 8r) = 0,$$

and therefore

$$\begin{aligned}\Sigma(y+z-x)^2(y-z)^2 &= \frac{1}{4}(2(p^2-4q)^2 + 6p(p^3-4pq+8r)) \\ &= 2p^4 - 10p^2q + 8q^2 + 12pr;\end{aligned}$$

also

$$\Sigma yz(y-z)^2 = p^2q + 3pr - 4q^2,$$

and the confirmation of (6) is trivial.

It is somewhat late in the day to point out that most if not all of the identities with which we have been dealing might have been discovered by application of the principle used above to identify the function ϕ in (10). For example, the only non-negative products of the fourth degree in x, y, z with the vanishing points of ϕ have one of the two forms $(aY+bZ)^4$, $cyz(y-z)^2$, where $X = (y-z)(y+z-x)$ and Y, Z are derived cyclically from X . Since $\Sigma X = 0$, $2\Sigma YZ = -\Sigma X^2$, and therefore $\Sigma(aY+bZ)^4 = (a^2-ab+b^2)\Sigma X^2$. It follows, since ϕ is symmetrical, that if $p\phi$ can be exposed in any way as the sum of non-negative products, there is an exposure in the form $A\Sigma X^2 + B\Sigma yz(y-z)^2$, and from this point the exposure of $p\phi$ is easily completed.

A more gratifying application of the principle is to the function ϕ itself. If ϕ can be expressed as a sum of non-negative products, the term x^3 in ϕ must come from one product or a number of products of the form $cxyu^2$, where u is linear in the three variables and is zero if $y=0$ and $x=z$, if $z=0$ and $x=y$, and if $x=y=z$. Since there is no such linear function, no exposure of ϕ is possible. Along similar lines it is easy to prove that the function $\phi(x^2, y^2, z^2)$, which is non-negative for all real values of x, y, z , can not be exposed as a sum of squares; there are ten coefficients in the general cubic homogeneous in three variables, and for a function of degree six I should hardly expect trial and failure to throw much light on the problem of exposure.

E. H. N.

2653. Summations by calculation of probabilities (Note 2584).

In note 2584 Dr. H. W. Haskey evaluates two sums by means of calculating probabilities. It seems to be worth while to point out that both of his results are tantamount to Vandermonde's theorem (with integral elements) in a somewhat disguised form. He quotes another writer as stating that the first result is obtainable "with some trouble if only elementary methods are used". In view of my assertion, this quotation is probably to be regarded as what Michael Finsbury would have described as a little judicious levity.

The notation which I use for products is

$$[\mu]_{[r]} = \mu(\mu-1)(\mu-2) \dots (\mu-r+1), \quad [\mu]_{[0]} = 1,$$

with r a positive integer, the square brackets being used to distinguish such products of descending factors from the products of ascending factors which are associated with hypergeometric series. With this notation Vandermonde's theorem assumes the form

$$\sum_{r=0}^m \binom{m}{r} [\mu]_{[m-r]} [\nu]_r = [\mu+\nu]_{[m]},$$

with m a positive integer (zero included) and μ, ν being subject to no restrictions.

We now consider Dr. Haskey's first summation, namely

$$\sum_r \binom{r+x}{x} \binom{Np}{x+r} \binom{Nq}{n-x-r} / \binom{N}{n} = \binom{n}{x} \binom{Np}{x} / \binom{N}{x},$$

in which $0 < p < 1$, $q = 1 - p$ and it is stated (or implied) in the course of his work that $x, n, n - x, Np - x, Nq - n + x$ are positive integers (zero included), and the sum runs from 0 to $n - x$.

We proceed to prove this result in the form

$$\sum_{r=0}^{n-x} \binom{r+x}{x} \binom{Np}{x+r} \binom{Nq}{n-x-r} = \binom{Np}{x} \binom{N-x}{n-x},$$

assuming only that $x, n, n - x$ are positive integers (zero included) and that $p + q = 1$; we place no restrictions at all on the numbers N and p .

We write the numerators of the binomial coefficients on the left in the form of products, and then we have

$$\begin{aligned} & \sum_{r=0}^{n-x} \binom{r+x}{x} \binom{Np}{x+r} \binom{Nq}{n-x-r} \\ &= \sum_{r=0}^{n-x} \frac{(r+x)!}{x! r!} \frac{[Np]_{(x+r)}}{(x+r)!} \frac{[Nq]_{(n-x-r)}}{(n-x-r)!} \\ &= \frac{[Np]_{(x)}}{x!(n-x)!} \sum_{r=0}^{n-x} \binom{n-x}{r} [Np-x]_{(r)} [Nq]_{(n-x-r)} \\ &= \frac{[Np]_{(x)}}{x!(n-x)!} [N-x]_{(n-x)} \\ &= \binom{Np}{x} \binom{N-x}{n-x}, \end{aligned}$$

by making some simple rearrangements of products and using Vandermonde's theorem. This is the result to be proved.

Next, provided that N is not one of the integers $0, 1, 2, \dots, n - 1$, we have

$$\begin{aligned} \binom{N-x}{n-x} &= \frac{[N-x]_{(n-x)}}{(n-x)!} = \frac{[N]_{(n)}}{[N]_{(x)} (n-x)!} \\ &= \binom{n}{x} \binom{N}{n} / \binom{N}{x}, \end{aligned}$$

whence Dr. Haskey's first result follows immediately.

Dr. Haskey's second summation is

$$\sum_r \frac{1}{r! s! t! u!} = \frac{n!}{(a+b)!(c+d)!(a+c)!(b+d)!},$$

where

$$r+s=a+b, \quad s+u=b+d,$$

$$r+t=a+c, \quad t+u=c+d,$$

the four numbers on the right are positive integers (zero included) and

$$n=a+b+c+d;$$

while the summation extends over all integral values of r for which none of the numbers r, s, t, u is a negative integer.

It is convenient to distinguish four cases, according to the signs of $a-d$ and $b-c$.

With $a \leq d$ and $b \leq c$ we write the sum on the left in the form

$$\begin{aligned} & \sum_{r=0}^{a+b} \frac{1}{r!(a+b-r)!(a+c-r)!(d-a+r)!} \\ &= \frac{1}{(a+b)!(a+c)!(b+d)!} \sum_{r=0}^{a+b} \binom{a+b}{r} [b+d]_{[a+b-r]} [a+c]_{[r]} \\ &= \frac{[n]_{[a+b]}}{(a+b)!(a+c)!(b+d)!}, \end{aligned}$$

by Vandermonde's theorem, and the required result follows at once.

With $a \leq d$ and $b \geq c$ we merely interchange b with c (and s with t) in the preceding piece of work, with the same result as before.

With $a \geq d$ we interchange a with d (and r with u) in the two preceding pieces of work, with the same result as before. This completes the investigation.

It is an interesting study to examine how it comes about that these four cases can coalesce into the single investigation given by Dr. Haskey. The reader may also find it interesting to see how far he can set up one-one correspondences between the steps of Dr. Haskey's two summations and the steps of the proof of Vandermonde's theorem given by Chrystal in his *Algebra*; Chrystal, instead of using the terminology of the theory of probability uses the equivalent terminology of combinatory analysis.

G. N. WATSON

2654. Two questions in the 1881 Tripos.

Is it not likely that "two cognate problems" appeared in the 1881 Tripos for a very simple reason, viz. that at the time concerned the solving of questions on the relations of the roots of equations was much in vogue? A scrutiny of the mathematical writings of the time might be very revealing. At any rate the following direct approach to the solution of the second problem would be easily within the compass of examinees well versed in dealing with functions of the roots of a cubic.

α, β, γ and $\alpha_1, \beta_1, \gamma_1$ are the roots of $x^3 - px^2 + qx - r = 0$, $x^3 - p_1x^2 + q_1x - r_1 = 0$. Let $\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = \lambda$, $\alpha\beta_1 + \beta\gamma_1 + \gamma\alpha_1 = \mu$, $\alpha\gamma_1 + \beta\alpha_1 + \gamma\beta_1 = \nu$.

$$\text{I. } \lambda + \mu + \nu = (\alpha + \beta + \gamma)(\alpha_1 + \beta_1 + \gamma_1) = pp_1.$$

$$\begin{aligned} \text{II. } \lambda^2 + \mu^2 + \nu^2 &= (\alpha^2 + \beta^2 + \gamma^2)(\alpha_1^2 + \beta_1^2 + \gamma_1^2) \\ &\quad + 2(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha_1\beta_1 + \beta_1\gamma_1 + \gamma_1\alpha_1) \\ &= (p^2 - 2q)(p_1^2 - 2q_1) + 2qq_1 \\ &= p^2p_1^2 - 2p^2q_1 - 2p_1^2q + 6qq_1 \end{aligned}$$

$$\begin{aligned} \text{Also } \lambda^2 + \mu^2 + \nu^2 &= (\lambda + \mu + \nu)^2 - 2(\mu\nu + \nu\lambda + \lambda\mu) \\ &= p^2p_1^2 - 2(\mu\nu + \nu\lambda + \lambda\mu) \end{aligned}$$

$$\therefore \mu\nu + \nu\lambda + \lambda\mu = p^2q_1 + p_1^2q - 3qq_1.$$

$$\text{III. } \begin{aligned} \lambda\mu\nu &= \alpha\beta\gamma \Sigma\alpha_1^3 + \alpha_1\beta_1\gamma_1 \Sigma\alpha^3 + 3\alpha\beta\gamma\alpha_1\beta_1\gamma_1 \\ &\quad + \Sigma\alpha^2\beta \cdot \Sigma\alpha_1^2\beta_1 + \Sigma\alpha\beta^2 \cdot \Sigma\alpha_1\beta_1^2. \end{aligned}$$

$$\begin{aligned} \text{But } \Sigma\alpha^3 &= p^3 - 3pq + 3r \\ \Sigma\alpha_1^3 &= p_1^3 - 3p_1q_1 + 3r_1 \\ 3\alpha\beta\gamma\alpha_1\beta_1\gamma_1 &= 3rr_1. \end{aligned}$$

So it remains only to find $\Sigma\alpha^2\beta$, $\Sigma\alpha\beta^2$, etc.

$$\begin{aligned} \text{Now } \Sigma\alpha^3\beta^3 &= (\Sigma\alpha\beta)^3 - 3\alpha\beta\gamma(\Sigma\alpha^2\beta + \Sigma\alpha\beta^2) - 6\alpha^2\beta^2\gamma^2 \\ &= q^3 - 3pqr + 3r^2. \end{aligned}$$

Also $\Sigma \alpha^2 \beta^2 = \Sigma \alpha^2 \beta \cdot \Sigma \alpha \beta^2 - \alpha \beta \gamma \Sigma \alpha^2 - 3 \alpha^2 \beta^2 \gamma^2$
 $= \Sigma \alpha^2 \beta \cdot \Sigma \alpha \beta^2 - (rp^2 - 3pqr + 6r^2)$

$\therefore \Sigma \alpha^2 \beta \cdot \Sigma \alpha \beta^2 = rp^2 + q^2 - 6rpq + 9r^2$
 But $\Sigma \alpha^2 \beta + \Sigma \alpha \beta^2 = pq - 3r$

$$\therefore \Sigma \alpha^2 \beta = \frac{1}{2}(pq - 3r) \pm \frac{1}{2}\sqrt{\Delta}$$

$$\Sigma \alpha \beta^2 = \frac{1}{2}(pq - 3r) \mp \frac{1}{2}\sqrt{\Delta}$$

where $\Delta = p^4 q^2 - 4q^3 - \{4rp^3 - 18(pqr - \frac{3}{2}r^2)\}$

N.B. $\frac{\partial \Delta}{\partial r} = -\{4p^3 - 18(pq - 3r)\}$

$$\therefore \lambda \mu \nu = \frac{1}{2}(2rp_1^2 + 2r_1 p^2 + pqp_1 q_1 - 9pqr_1 - 9p_1 q_1 r + 27rr_1 \pm \sqrt{4\Delta_1})$$

Hence the equation whose roots are λ, μ, ν is

$$x^3 - pp_1 x^2 + (p^2 q_1 + p_1^2 q - 3qq_1)x - \lambda \mu \nu = 0$$

$$\text{Let } x = y + \frac{1}{3}pp_1.$$

Then the equation is :

$$y^3 - \frac{1}{3}(p^2 - 3q)(p_1^2 - 3q_1)y - \frac{1}{216}(4p^3 - 18(pq - 3r))(4p_1^3 - 18(p_1 q_1 - 3r_1)) = \pm \frac{1}{2}\sqrt{4\Delta_1},$$

$$\text{i.e. } (x - \frac{1}{3}pp_1)^3 - \frac{1}{3}(x - \frac{1}{3}pp_1)(p^2 - 3q)(p_1^2 - 3q_1) = \frac{1}{216} \frac{\partial \Delta}{\partial r} \cdot \frac{\partial \Delta_1}{\partial r_1} \pm \frac{1}{2}\sqrt{4\Delta_1}$$

A problem of much greater difficulty is the following :

α, β, γ and $\alpha_1, \beta_1, \gamma_1$ are the roots of the cubics $x^3 - px^2 + qx - r = 0$ and $x^3 - p_1 x^2 + q_1 x - r_1 = 0$.

If

$$\begin{aligned} \alpha^2 \alpha_1^2 + \beta^2 \beta_1^2 + \gamma^2 \gamma_1^2 &= \lambda \\ \alpha^2 \beta_1^2 + \beta^2 \gamma_1^2 + \gamma^2 \alpha_1^2 &= \mu \\ \alpha^2 \gamma_1^2 + \beta^2 \alpha_1^2 + \gamma^2 \beta_1^2 &= \nu \end{aligned}$$

prove that the equation whose roots are λ, μ, ν is

$$(x - A)^3 - \frac{1}{3}(x - A)B = \frac{1}{54} C \pm \frac{1}{2}\sqrt{4\Delta_1}$$

where

$$\begin{aligned} A &= \frac{1}{3}(p^2 - 2q)(p_1^2 - 2q_1) \\ B &= \{(p^2 - 2q)^2 - 3(q^2 - 2rp)\} \cdot \{(p_1^2 - 2q_1)^2 - 3(q_1^2 - 2r_1 p_1)\} \\ C &= \{2(p^2 - 2q)^3 - 9(p^2 - 2q)(q^2 - 2rp) + 27r^2\} \\ &\quad \times \{2(p_1^2 - 2q_1)^3 - 9(p_1^2 - 2q_1)(q_1^2 - 2r_1 p_1) + 27r_1^2\} \end{aligned}$$

and

$$\begin{aligned} \Delta &= \{(p^2 - 2q)(q^2 - 2rp) - 3r^2\}^2 \\ &\quad - 4[(q^3 - 3pqr + 3r^2)^2 - 2r^3(p^3 - 3pq + 3r) \\ &\quad + r^2\{(p^2 - 2q)^3 - 3(p^2 - 2q)(q^2 - 2rp) + 6r^2\}] \end{aligned}$$

These are particular cases of a general problem which I now state and prove.
Problem.

α, β, γ and $\alpha_1, \beta_1, \gamma_1$ are the roots of $x^3 - px^2 + qx - r = 0, x^3 - p_1 x^2 + q_1 x - r_1 = 0$.

If

$$\begin{aligned} \alpha^n \alpha_1^n + \beta^n \beta_1^n + \gamma^n \gamma_1^n &= \lambda \\ \alpha^n \beta_1^n + \beta^n \gamma_1^n + \gamma^n \alpha_1^n &= \mu \\ \alpha^n \gamma_1^n + \beta^n \alpha_1^n + \gamma^n \beta_1^n &= \nu \end{aligned}$$

n being any integer (positive, zero, or negative), then the equation whose roots are λ, μ, ν is

$$(x - \frac{1}{3}A)^2 - \frac{1}{3}(x - \frac{1}{3}A)B = \frac{1}{8}C \pm \frac{1}{3}\sqrt{4A_1}$$

where

$$\begin{aligned} A &= \Sigma x^n \cdot \Sigma x_1^{-n} \\ B &= \{(\Sigma x^n)^2 - 3\Sigma x^n \beta^n\} \{(\Sigma x_1^{-n})^2 - 3\Sigma x_1^{-n} \beta_1^{-n}\} \\ C &= \{2(\Sigma x^n)^3 - 9\Sigma x^n \Sigma x_1^{-n} \beta^n + 27r^n\} \{2(\Sigma x_1^{-n})^3 - 9\Sigma x_1^{-n} \Sigma x_1^{-n} \beta_1^{-n} + 27r_1^{-n}\} \\ D &= (\Sigma x^n \Sigma x_1^{-n} \beta^n + 9r^n)^2 - 4(r^n)(\Sigma x^n)^3 + (\Sigma x^n \beta^n)^3 + 27r^{2n} \\ \Delta_1 &= (\Sigma x_1^{-n} \Sigma x_1^{-n} \beta_1^{-n} + 9r_1^{-n})^2 - 4(r_1^{-n})(\Sigma x_1^{-n})^3 + (\Sigma x_1^{-n} \beta_1^{-n})^3 + 27r_1^{-2n} \end{aligned}$$

Proof.

The required equation is $x^3 - x^2 \Sigma \lambda + x \Sigma \mu\nu - \lambda \mu\nu = 0$. On removing the second term and replacing, it becomes

$$(x - \frac{1}{3}\Sigma\lambda)^3 - \frac{1}{3}(x - \frac{1}{3}\Sigma\lambda)\{\Sigma(\lambda)^2 - 3\Sigma_{\mu\nu}\} = \frac{1}{54}\{4(\Sigma\lambda)^3 - 18\Sigma\lambda \cdot \Sigma_{\mu\nu} + 54\lambda_{\mu\nu}\}$$

$$\begin{aligned} \text{I.} \quad & \Sigma \lambda = \Sigma \alpha^n \cdot \Sigma \alpha_1^{-n} \\ \text{Since} \quad & \lambda^2 + \mu^2 + \nu^2 = \Sigma \alpha^{2n} \Sigma \alpha_1^{-2n} + 2 \Sigma \alpha^n \beta^n \cdot \Sigma \alpha_1^{-n} \beta_1^{-n} \\ \text{and} \quad & \lambda^2 + \mu^2 + \nu^2 = (\Sigma \alpha^n)^2 \cdot (\Sigma \alpha_1^{-n})^2 - 2 \Sigma \mu \nu \\ & \therefore \Sigma \mu \nu = \frac{1}{2} (\Sigma \alpha^n)^2 (\Sigma \alpha_1^{-n})^2 - \frac{1}{2} \Sigma \alpha^{2n} \Sigma \alpha_1^{-2n} - \Sigma \alpha^n \beta^n \cdot \Sigma \alpha_1^{-n} \beta_1^{-n}. \\ \text{But} \quad & \Sigma \alpha^{2n} = (\Sigma \alpha^n)^2 - 2 \Sigma \alpha^n \beta^n, \text{ etc.} \end{aligned}$$

$$\text{II. } \quad \therefore \Sigma \mu \nu = (\Sigma \alpha^n)^2 \Sigma \alpha_1^n \beta_1^n + (\Sigma \alpha_1^n)^2 \Sigma \alpha^n \beta^n - 3 \Sigma \alpha^n \beta^n \cdot \Sigma \alpha_1^n \beta_1^n.$$

$$\text{Again } \quad \therefore (\Sigma \lambda)^2 - 3 \Sigma_{\mu\nu} = \{(\Sigma \alpha^n)^2 - 3 \Sigma \alpha^n \beta^n\} \{(\Sigma \alpha_1^n)^2 - 3 \Sigma \alpha_1^n \beta_1^n\},$$

$$\lambda_{\mu\nu} = r^n \Sigma \alpha_1^{2n} + r_1^n \Sigma \alpha^{2n} + 3r^n r_1^n$$

$$+ \{(\Sigma \alpha^{2n})^2 - 3 \Sigma \alpha^{2n} \beta^{2n}\} \cdot \Sigma \alpha_1^{2n} \beta_1^{2n} + \Sigma \alpha^n \beta^{2n} \cdot \Sigma \alpha_1^n \beta_1^{2n}.$$

But $\Sigma \alpha^{2n} = (\Sigma \alpha^n)^2 - 3 \Sigma \alpha^n \beta^n + 3r^n$, etc.

$$\text{IV. } \therefore \lambda_{\mu\nu} = r^n (\Sigma \alpha_1^n)^2 + r_1^{-n} (\Sigma \alpha^n)^2 - 3r^n \Sigma \alpha_1^n \Sigma \alpha_1^n \beta_1^{-n} - 3r_1^{-n} \Sigma \alpha^n \Sigma \alpha^n \beta^n + 9r^n r_1^{-n} + (\Sigma \alpha^n \beta^n \cdot \Sigma \alpha_1^{-n} \beta_1^{-n} + \Sigma x^n \beta^{2n} \cdot \Sigma \alpha_1^{-n} \beta_1^{-2n}).$$

The value of the last bracket is obtained thus :

$$\text{Again } \Sigma \alpha^{3n} \beta^{3n} = (\Sigma \alpha^n \beta^n)^3 - 3r^n \Sigma \alpha^n \Sigma \alpha^n \beta^n + 3r^{3n}$$

$$\text{Also } \Sigma \alpha^{2n} \beta^{2n} = \Sigma \alpha^{2n} \beta^{2n} - r^n (\Sigma \alpha^n)^2 + 3r^n \Sigma \alpha^n \Sigma \beta^n - 6r^{2n}$$

Hence from (1) and (2)

$$\Sigma x^{2n} \beta^n - \Sigma x^n \beta^{2n} = \pm [(\Sigma x^n \Sigma x^n \beta^n + 9r^n)^2 - 4\{r^n (\Sigma x^n)^2 + (\Sigma x^n \beta^n)^2 + 27r^{2n}\}]^{1/2}$$

$$\therefore \Sigma x^{2n} \beta^n = \frac{1}{2} (\Sigma x^n \cdot \Sigma x^n \beta^n - 3r^n) \pm \frac{1}{2} \sqrt{\Delta}$$

$$V. \quad \therefore \Sigma x^{2n} \beta^n = \Sigma x_1^{2n} \beta_1^n + \Sigma x^\alpha \beta^{2n} = \Sigma x_1^n \beta_1^{2n}$$

$$= \frac{1}{2} (\Sigma x^\alpha \Sigma x_1^n \beta^n - 3r^n) (\Sigma x_1^n \Sigma x_1^n \beta_1^n - 3r_1^n) \pm \frac{1}{2} \sqrt{4\Delta_1}.$$

$$\text{VI. } \therefore 54\lambda_{\mu\nu} = 54r^n(\Sigma\alpha_1^n)^2 + 54r_1^n(\Sigma\alpha^n)^2 \\ + 27\Sigma\alpha^n\Sigma\alpha^n\Sigma\alpha^n\beta^n\Sigma\alpha_1^n\beta_1^n - 243r^n\Sigma\alpha_1^n\Sigma\alpha_1^n\beta_1^n \\ - 243r_1^n\Sigma\alpha^n\Sigma\alpha^n\beta^n + 729r^n r_1^n + 27\sqrt{4A}.$$

VII. Hence

$$\begin{aligned} 4(\Sigma\lambda)^3 - 18\Sigma\lambda \cdot \Sigma\mu\nu + 54\lambda\mu\nu \\ = 4(\Sigma\alpha^n)^3 (\Sigma\alpha_1^n)^3 - 18(\Sigma\alpha^n)^3 \Sigma\alpha_1^n \Sigma\alpha_1^n \beta_1^n - 18(\Sigma\alpha_1^n)^3 \Sigma\alpha^n \Sigma\alpha^n \beta^n \\ + 81\Sigma\alpha^n \Sigma\alpha_1^n \Sigma\alpha^n \beta^n \Sigma\alpha_1^n \beta_1^n + 54r^n(\Sigma\alpha_1^n)^3 + 54r_1^n(\Sigma\alpha^n)^3 \\ - 243r^n \Sigma\alpha_1^n \Sigma\alpha_1^n \beta_1^n - 243r_1^n \Sigma\alpha^n \Sigma\alpha^n \beta^n + 729r^n r_1^n \pm 27\sqrt{4A_1} \\ = (2(\Sigma\alpha^n)^3 - 9\Sigma\alpha^n \Sigma\alpha^n \beta^n + 27r^n)(2(\Sigma\alpha_1^n)^3 - 9\Sigma\alpha_1^n \Sigma\alpha_1^n \beta_1^n + 27r_1^n) \\ \pm 27\sqrt{4A_1}. \end{aligned}$$

Therefore the required equation is :

$$(x - \frac{1}{3}A)^3 - \frac{1}{3}(x - \frac{1}{3}A)B = \frac{1}{3}C \pm \frac{1}{2}\sqrt{4A_1}$$

where A , B , C , A_1 , and Δ_1 are as stated.

N.B.—It may be noted that successive values of $\Sigma\alpha^n$ and $\Sigma\alpha^n\beta^n$ are readily obtained from the relations :

$$\begin{aligned} (1) \quad \Sigma\alpha^n &= p\Sigma\alpha^{n-1} - q\Sigma\alpha^{n-2} + r\Sigma\alpha^{n-3} \\ (2) \quad \Sigma\alpha^n\beta^n &= q\Sigma\alpha^{n-1}\beta^{n-1} - rp\Sigma\alpha^{n-2}\beta^{n-2} + r^2\Sigma\alpha^{n-3}\beta^{n-3}. \end{aligned}$$

For speedy evaluation we have :

$$\begin{aligned} (1) \quad \Sigma\alpha^{-3} &= \frac{q^3 - 3qr^2 + 3r^2}{r^3} \\ \Sigma\alpha^{-2} &= \frac{q^2 - 2rp}{r^2} \\ \Sigma\alpha^{-1} &= \frac{q}{r} \\ \Sigma\alpha^0 &= 3 \\ \Sigma\alpha &= p \\ \Sigma\alpha^2 &= p^2 - 2q \\ \Sigma\alpha^3 &= p^3 - 3pq + 3r, \text{ etc.} \end{aligned}$$

and the relation is

$$\begin{aligned} (2) \quad \Sigma\alpha^{-3}\beta^{-3} &= \frac{p^3 - 3pq + 3r}{r^3} \\ \Sigma\alpha^{-2}\beta^{-2} &= \frac{p^2 - 2q}{r^2} \\ \Sigma\alpha^{-1}\beta^{-1} &= \frac{p}{r} \\ \Sigma\alpha^0\beta^0 &= 3 \\ \Sigma\alpha\beta &= q \\ \Sigma\alpha^2\beta^2 &= q^2 - 2rp \\ \Sigma\alpha^3\beta^3 &= q^3 - 3qr^2 + 3r^2, \text{ etc.} \end{aligned}$$

and the relation is

$$\Sigma\alpha^n\beta^n = q\Sigma\alpha^{n-1}\beta^{n-1} - rp\Sigma\alpha^{n-2}\beta^{n-2} + r^2\Sigma\alpha^{n-3}\beta^{n-3}.$$

NEIL Y. WILSON

2655. A note on Notes 2548 and 2566.

It seems to have been overlooked that the substance of the above Notes is contained in P. Cohn's Note 1644 (XXVII) and my comment thereon, Note 1726 (XXVIII).

A. R. PARGETER

2656. On Note 2526.

Another proof by projection runs as follows:

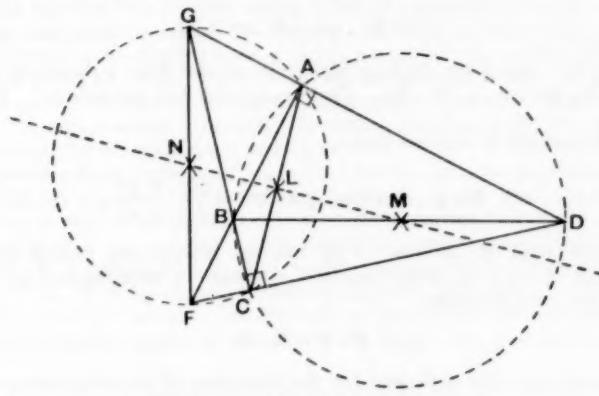


FIG. 1.

A general quadrilateral with opposite vertices A, C ; B, D ; F, G is affine equivalent to one which is right-angled at A, C . But the collinearity of the mid-points L, M, N of AC, BD, FG for a quadrilateral of this special type follows at once from a consideration of the circles on BD and FG as diameters (see figure). The result is therefore true generally since an affine transformation preserves mid-points of segments. HAZEL PERFECT

HAZEL PERFECT

2657. A vector treatment of the Pfaffian in three variables.

In the *Mathematical Gazette*, Vol. 37, No. 320, there is a note by F. E. Relton on the necessary condition that a Pfaffian in three variables possesses an integrating factor. He used the well-known vector derivation, and suggested it would be interesting to have a vector proof of the sufficiency of the condition. Such a proof is contained in the following.

Consider the Pfaffian

$$X \, dx + Y \, dy + Z \, dz = R \, dr,$$

where \mathbf{R}, \mathbf{r} are the vectors $X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, respectively. The first step is to reduce the Pfaffian to its canonical form, namely

$$\mathbf{R} \cdot d\mathbf{r} = du + v dw = (\nabla u + v \nabla w) \cdot d\mathbf{r},$$

where u, v, w are functions of x, y, z . This reduction can be done as follows.
The forms are equivalent if

Now

so that v and w both satisfy the linear equation

Let $\alpha(r) = \text{const.}$, and $\beta(r) = \text{const.}$, be two independent integrals of the subsidiary equations of (3), namely

$$d\mathbf{r} = \lambda(\nabla \wedge \mathbf{R}),$$

where λ is a constant, then v and w can be taken to be any two independent functions of α and β .

From (1) and (2) u satisfies

$$(\nabla \wedge \mathbf{R}) \cdot \nabla u = \mathbf{R} \cdot (\nabla \wedge \mathbf{R}), \quad \dots \dots \dots \quad (4)$$

which can be solved by finding integrals of the four subsidiary equations $dr = \lambda(\Delta \wedge \mathbf{R})$, $du = \lambda(\mathbf{R} \cdot \nabla \wedge \mathbf{R})$. This completes the reduction to the canonical form.

From (1) and (2) it follows that

$$\mathbf{R} \cdot \nabla \wedge \mathbf{R} = \nabla u \cdot \nabla v \wedge \nabla w = \frac{\partial(u, v, w)}{\partial(x, y, z)}. \quad \dots \dots \dots \quad (5)$$

Suppose now that $\mathbf{R} \cdot d\mathbf{r} = du + v dw$ has an integrating factor and can be written in the form $\lambda d\phi$, then clearly u, v, w cannot be independent functions. From (5) this requires that

$$\mathbf{R} \cdot \nabla \wedge \mathbf{R} = 0. \quad \dots \dots \dots \quad (6)$$

This equation is also sufficient for the existence of an integrating factor, for if it holds (5) shows that a relation exists between u, v and w , say $v = F(u, w)$.

Then

$$\mathbf{R} \cdot d\mathbf{r} = du + F(u, w)dw.$$

The right-hand side is now a Pfaffian in two variables, and such a Pfaffian is known to have an infinite number of integrating factors.

University of Sydney, Australia.

L. C. Woods

2658. The Euclidean quadratic.

Among the diminishing band of those who read Euclid at school there may be some who have kept their text books and can confirm the omission of the twenty-seventh, twenty-eighth and twenty-ninth propositions of the Sixth Book; in one edition it is curiously observed that they are "not read", in another they are relegated to obscurity as "cumbrous in form and of little value as geometrical results" but Simson* had long ago remarked on their importance and Heath† has stressed this in these words:

"In the vital propositions 27, 28, 29 the Pythagorean application of areas appears in its most general form, equivalent to the geometrical solution of the most general form of quadratic equation where that equation has a real and positive root. The method is fundamental in Greek geometry; it is, for instance, the foundation of Euclid's Book X (on irrationals) and the whole treatment of conic sections by Apollonius of Perga."

The equations that Euclid solves in VI.28 and 29 are these:

VI.28

$$ax^2 \sin \theta + c = bax \sin \theta$$

VI.29

$$ax^2 \sin \theta + bax \sin \theta = c$$

* "The unique status of Euclid as a text book in England . . . may be said to date from the publication of Robert Simson's *Elements of Euclid*, which first appeared in Latin and in English in 1756" (Heath).

† *A Manual of Greek Mathematics* by Sir Thomas L. Heath, p. 231 (1931).

The data include the coefficient b as a finite straight line and a rectilinear figure representing the area to which the parallelogram c must be made equal. The coefficient a is expressed in the solution by the ratio (ax/x) and in the data by the ratio of the corresponding sides of a parallelogram to which that containing the solution must be similar. The constructional problem in VI.28, therefore, is to apply to the line b a parallelogram of the required area and of such form as to be "deficient" (in covering the line) by a parallelogrammic figure (containing the solution) similar to a given one. In VI.29 the parallelogram c must "exceed" (overlap) the line b by the amount ax^2 in the required form symbolised here by θ .

Euclid having been dethroned, probably few would now be interested in the full detail of his process: let it suffice, therefore, to say that the first step is to construct on half the line a a parallelogram similar and similarly situated to the given one (VI.18). If the parallelogram so constructed happens also to be equal to the given area then that which was enjoined will have been done but if not then let it be greater* and let another be constructed (elsewhere) that is both equal and similar (VI.25) and let this be redrawn within the larger one. The complete figure of construction forms the parallelogram c and cuts off x .

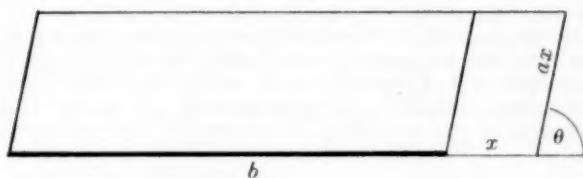
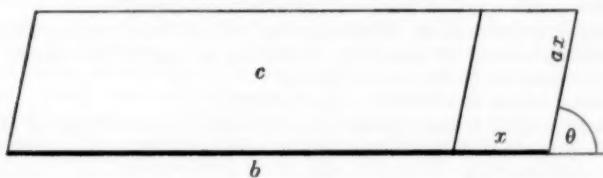


FIG. 1.

Euclid justifies his procedure by references to I.36, 43 and VI.18, 21, 24, 25, 26. Of these VI.18 and 25 are the most important in the sense that they are constructively related to the "application of areas" but the prototype in this field is I.44 which applies to a given straight line a parallelogram equal to a given triangle and having an angle equal to a given angle. This proposition, says Heath, "will always remain one of the most impressive in all geometry when account is taken of the great importance of the result obtained and . . . the simplicity of the means employed".

Of course, it must not be supposed that the omission of VI.27-29 implied any general lack of attention in school texts to the algebraic aspects of Euclid, but the omission is all the more remarkable on that account and particularly

* In VI.28 it is a necessary condition (covered by the enunciation) that the given rectilineal figure must not be greater than the parallelogram described on half the line and similar to the defect. This follows from VI.27 which is a generalised proof of the particular fact that the square on half the line is the largest rectangle that can be constructed from any two parts of the line.

so when proposition VI.29 is considered as a constructive aspect of theorem II.6 :

II.6. If a straight line b is bisected and extended by x , the rectangle c contained by the whole line thus extended and the extended part, together with the square on half the line bisected, is equal to the square Z on the line made up of the half and the part produced.

$$c = x(x+b) \quad \therefore x^2 + bx = c; \quad Z = c + (b/2)^2 = x^2 + bx + (b/2)^2 = (x + (b/2))^2 \\ \therefore x + (b/2) = \sqrt{Z} \quad \therefore x = \sqrt{Z} - (b/2).$$

A. E. BERRIMAN

2659. Approximate construction of regular polygons.

Mr. Banner in Note 2297 (XXXVI, 1952, No. 318, p. 276) and Mr. Williamson in Note 2508 (XXXIX, 1955, No. 328, p. 134) have given approximate constructions of the regular heptagon. In *Mathesis*, 1935, p. 179, I had already mentioned Mr. Banner's construction.

The side of the regular heptagon inscribed in a circle with radius unit is $0.86776\dots$ and Mr. Banner's construction leads to $0.86602\dots$ (error $e < 0.0018$) as Mr. Williamson's method leads to $0.86638\dots$ ($e < 0.0015$).

In my paper published in *Mathesis*, loc. cit., I have shown that a more accurate construction is obtained by producing the side of the regular inscribed decagon by one quarter of the circumradius.

The obtained value is $0.86803\dots$ ($e < 0.0003$).

In the same paper, I have given approximate constructions of the side c_n of the regular n -sided inscribed polygon up to $n=31$; they are based on well-known constructions of c_n for values of n for which the construction is possible by ruler and compasses. For instance, c_{13} is approximately the quarter of the segment obtained by producing the side of the inscribed square by half the circumradius ($e < 0.00008$). Further, c_{17} will be obtained by reducing the side of the regular inscribed decagon by one quarter of the radius ($e < 0.0006$); so, approximately, $2c_{10} = c_7 + c_{17}$. The here given construction for c_{17} is more accurate than the one indicated by Rouché et de Comberousse, *Traité de Géométrie*, ed. 1929, Vol. I, p. 424, based on the fact that approximately $c_{17} = c_8 - c_6$ ($e < 0.002$). Finally c_{25} is approximately one quarter of the circumradius ($e < 0.0007$).

R. GOORMAGHTIGH

2660. On notes 1617, 2297 and 2508.

Since I believe I started this heptagon on its travels, perhaps I may be permitted to add a note. My original construction (Note 1617, XXVI, 1942, p. 180) gave, for the angle subtended by the side of the heptagon at the centre, $51^\circ 38'$. The correct value is $51^\circ 25.7'$. The error in this construction is thus comparatively large compared with Mr. Banner's and Mr. Williamson's. A slight modification of my construction, which is also, I believe, well-known to draughtsmen yields a much closer result. On a diameter AB describe an equilateral triangle ABC . From AB cut off $AE = 2AB/7$ ($2AB/n$ for an n -gon) as before. Join CE and produce it to meet the circle at F . AF is an approximate side of a regular heptagon inscribed in the circle. The length of AF with unit radius is 0.8691 , so that the error is < 0.0014 . This construction is thus more accurate than Mr. Banner's or Mr. Williamson's, but not than Mr. Goormaghtigh's. It has, however, the merit of simplicity. Incidentally, the method is also more accurate than my original (in which $OC = 1\frac{1}{7}r$, instead of $r\sqrt{3}$) for the polygons with 8, 9, and 10 sides, but worse for those with 5, 11 and 12. It is completely accurate for polygons with 3, 4, and 6 sides, when of course it would not be used.

H. MARTYN CUNDY

REVIEWS

Mémoire sur la dérivation et son calcul inverse. By A. DENJOY. Published with the assistance of the Centre National de la Recherche Scientifique. Pp. vi, 380. 2700 fr. 1955. (Gauthier-Villars, Paris)

This volume must be classed as a source-book in the domain of modern advanced analysis. It is a phototype reproduction of the four famous wartime memoirs in which Denjoy solved the problem of finding the primitive function of any function known to be a derivative.

The process invented by Denjoy, which he called *totalisation* and based on the Riemann and Lebesgue integration procedures, had been announced in two communications to the Paris *Comptes-rendus* in 1912. The expanded work appeared successively in *Journ. de math. pures et appl.* (7^e série) I, 1915 (pp. 1-248), *Bull. Soc. Math. Fr.* 43, 1915 (pp. 161-248), *Ann. Ec. Norm.* (3), 33, 1916 (pp. 127-222), and *ibid.* 34, 1917 (pp. 181-236), and widens the scope of the process to apply to functions not known *a priori* to be derivatives.

In the interval, his idea had been taken up in a number of writings, notably in the field of trigonometric series which is still the most fertile for applications. He refers to them in a footnote as conferring a great deal of honour on a modest idea which chance had led him to be the first to enunciate (p. 344 in the continuous pagination superimposed on the original numbering). From a historical point of view, a list of these collateral writings, and of the main subsequent ones, would have greatly added to the value of the present republication. Only the author's own works in the series are listed, and with the table of misprints and general table of contents at the end supplement what could be gleaned by consulting the original periodicals.

But Denjoy has prefaced the work with a new *Avertissement* which shows that he is far from passively wishing it to stand among the great documents of human thought as here advocated. He is profoundly concerned about the gulf existing between classical Analysis with its continuous functions, differentiable as often as may be required, with its well-behaved range of variables, on the one hand, and on the other the theory of abstract spaces, collections often of totally unconnected elements subject to conditions of an arbitrary nature. The transition between these two conceptual worlds could be provided for the student by a personal experience of the vagaries of real linear sets of points, the original jumping-off ground of the axiomatists. In this cause, he has heard say, a study of his work has often helped. So he hopes that in the new guise it may do so again.

This modest hope is consistent with the wider historic interest ; and if it is fulfilled, this will be as much due to the pleasing qualities of the presentation as to the subject-matter. The work has all the charm of one of the great nature-studies, such as Fabre's *Souvenirs entomologiques*, with its beauty and elegance of method and of style, the minute and careful analysis and descriptive power, and finally the sheer unstressed brilliance of the results achieved, starting here from the minimum that a candidate for the *licence* might be assumed to know in 1914.

At the same time, there is a fascination in realising that the beauty and elegance of Denjoy's methods were precisely what at first halted advance beyond the one-dimensional field, and reinforced the case for the budding axiomatists. Paradoxical as it may seem, the lack of clear-cut direct analogues in more dimensions for the properties he discovered and used with such skill, while it provided a most happy twist for the view that "Human nature abhors the *n*th dimension", only hastened the final twist, to the acceptance of a dimensionless axiomatic world, thus substituting for the mental effort formerly shirked one positively fearsome to the old school. This substitution, with the inevitable shift in intellectual class-consciousness, did not shake

Denjoy's original allegiance, which was never consciously restricted. He witnessed with the greatest sympathy the achievements both in the n -dimensional and in the abstract fields. The opening words of his work : "The object of the present Memoir is the study of the derivates of the most general continuous functions", omitting for twenty pages any explicit mention of the restriction to one variable, reveal his instinctive thought to be one-dimensional. But a remark on p. 20, and one or two other passages, concerning sets of points in space of more dimensions, show he had no intention of being cut off from the vanguard in any advance. This gallant attitude persists into this year of his scientific Jubilee. One feels, nevertheless, that his personal, active affection remains with the minds (now of the "depressed" classes) which need "concrete" visualisation as a start. Clearly he believes that the same delight and mental freedom can be theirs as he attained in his youth by the meticulous discipline embodied in this work.

R. C. H. YOUNG

Hydrodynamics. By G. BIRKHOFF. Rep. Pp. xiii, 186. \$1.75; cloth, \$3.50. 1955.

Non-Euclidean geometry. By R. BONOLA. Rep. With G. B. Halsted's translations of N. Lobachevski, "The theory of parallels" and J. Bolyai, "The science of absolute space". Pp. xii, 268, xxx, 71, 50. \$1.90; cloth, \$3.95. 1955.

Theory of groups of finite order. By W. BURNSIDE. 2nd edition, rep. Pp. xxiv, 512. \$2; cloth, \$3.95. 1955. (Dover Publications, New York).

Garrett Birkhoff's stimulating book appeared as recently as 1950. The author is concerned with the role of mathematics in bridging the gap between hydrodynamic theory and experiment, exposed by certain notorious paradoxes, and in establishing the foundations of dimensional analysis and of the theory of similitude and models. The volume is an admirable, even provocative, supplement to the classic treatises of hydrodynamic theory.

Bonola's introduction to non-euclidean geometry, in Carslaw's translation, is of permanent value because of its critical study of the 5th postulate from Euclid to Riemann and Clifford; the worth of this reprint is enhanced by the inclusion of translations of the classics by Lobachevski and Bolyai.

The second (1911) edition of Burnside offered the considered exposition of one of the great masters of group theory; since then very much has been done to simplify and generalise the theory, and to develop the field of application. Thus Burnside has become a classic which perhaps no one need read; but its firm substance and masterly, authoritative style remain a memorial to the author, whose reputation as an outstanding teacher is not yet forgotten in the Royal Naval College.

Altogether, three more excellent reprints in the Dover Company's lengthening list.

T. A. A. B.

Mathematics Today. By E. E. BIGGS and H. E. VIDAL. Introductory Course: Part 1, The Tools of Mathematics. Pp. 336, 7s. 6d. Part 2, Mathematics in Action. Pp. 288. 6s. 6d. Answers to each part 1s. 6d. 1955. (Ginn)

These volumes together with the three volumes previously published on Mathematics Today are intended by the authors to form a complete school course for the Alternative Syllabus. This makes it possible, for the first time, to assess the work as a whole from that angle. For convenience, the reviewer will refer to the parts as 1 to 5. When he reviewed part "3" he pointed out, at the time, how illogical it was to start in the middle of a course. The work was not conceived originally as a whole for, at the time, the authors clearly

had no intention of writing parts "1" and "2". The writing of parts "3" to "5" before "1" and "2" would not have mattered so much had the work been planned as a whole. In the course of time, parts "1" and "2" would have fitted correctly into place. As it is, there is no sense of order; topics are jumbled up and appear in most unexpected places. Frequently, a topic which has been dealt with in one part is repeated from the beginning in another part as if the first treatment did not exist. Thus in part "1" in the first chapter on Algebra, the formula is introduced and developed in detail from $A = lb$. In part "3" the formula is again introduced and developed from $A = lb$. Both treatments contain numerous examples on generalization and formulation. The first year treatment in part "3" is quite incongruous but this is one of several such instances. In part "1" we have a detailed treatment of areas, walls of a room and the like. This together with the walls of a room is all repeated in part "3". In both parts it is shown that the area of a triangle is $\frac{1}{2}bh$ and we get similar duplications with the parallelogram and the trapezium. In part "1" we have a full treatment of first year graph and many examples are set. In part "3" we have an even fuller treatment of first year graphs and even more examples are set. Other instances of such duplication are, introduction to plan and elevation (parts "2", "4"); ratio of areas and volumes of similar figures ("2", "4"); "proof" of the area of a circle by rearranging sectors ("2", "3"); introductory explanation of Pythagoras ("2", "3"); proofs of angle properties of a circle ("2", "3"); introductory work on the tangent of an angle ("2", "4"); introduction to symmetry ("1", "3"). It will be clear from these examples that the whole arrangement is very much a patchwork affair and that there is considerable looseness in planning. There is sometimes a lack of balance. Occasionally a particular type of example is flogged almost to death while another, an important type, will be omitted or receive scant treatment. For example, there are over 120 exercises on changing the subject of a formula but these contain only 8 examples of the important type illustrated by "Change the subject of $a = \frac{2+b}{5+b}$ to b ." In part 2 there are 316 exercises on percentages but these include none of the type "Given the S.P. and the profit, find the C.P." But then, we have the surprising arrangement that this chapter precedes the one on proportion to which it is subsidiary.

There are but six chapters in each of the first two parts (chapter 5 has 116 pages) and, as in the other parts, the chapter headings are sometimes arbitrary and meaningless. One disadvantage of this arrangement is that it is very difficult to open the book to a particular topic. Although many of the topics are mentioned in the table of contents, no page numbers are given and there is no index at the end. Where, for instance, is one to find "angles of a polygon"? Many teachers would like to know where to find the treatment of prime numbers, L.C.M. and H.C.F.; they will search in vain.

In spite of these criticisms, the work has a liveliness and freshness which many teachers will find very attractive. The authors have tried to introduce as much variety as possible into their problems and in this they have been very successful. They have avoided the dry "sum" type of example and have substituted live everyday illustrations. They have put in a good deal of thought and hard painstaking efforts to carry out these principles.

The treatment of Geometry is the least satisfactory part of the course. When the Alternative Syllabus was formulated, it was proposed that pupils be relieved of the burden of having to memorize the proofs of nearly all the theorems. It was never intended that this easement should imply the elimination of all logical structure. A great burden had been lifted from the shoulders of teachers and pupils alike. There was therefore no excuse whatever for scamping what was left of the course. There was always the risk that certain

teachers would not play the game and would scamp work on which examination questions would not be set. It might, at least, have been expected that authors of text books would not encourage this shoddy attitude (and it must be borne in mind what influence for good or bad authors have). In all the other text books on the Alternative Syllabus which I have seen, the authors have kept very well to that spirit of geometry which is characteristic of English teaching of the subject and which is so indispensable to a good education in mathematics. In the books under review, this spirit is markedly absent. Apart from the proofs of the examination theorems, there are few formal proofs. There is no logical structural development. Order is lacking and one is conscious of a jumble. The proof of the angle sum of a triangle is stuck in the middle of the treatment of angle properties of the circle. Loci and the loci theorems are put at the end of the Geometry Course where their impact on the subject is completely lost. Nowhere do we get a definition of parallel lines. Everywhere, there is a lack of precision and incisiveness. The essential facts of isosceles triangles are not mentioned, let alone proved, and there is no clear statement of either theorem on the angles of a polygon. The intercept theorem is "proved" by drawing transversals on ruled paper and measuring the intercepts; from this is "deduced" the first mid point theorem. Symmetry is not defined nor are the essential properties stated and yet it is used to give a two line "proof" that the tangent is perpendicular to the radius at the point of contact.

Part of the blame for this treatment is due to a weakness in the Jeffrey Report on the Alternative Syllabus. The "deletion of the proofs of theorems" left a vacuum which should have been filled. Some statement however brief, should have been added defining clearly the principles which should guide the development and teaching of the subject. This applies also to the statements of the syllabuses of Examining Bodies. The absence of such statements has inevitably given rise to ambiguities and misinterpretations in the minds of many teachers. In fact, the situation is quite serious and dangerous. Till now, the quality of English geometry teaching has been second to that of no country. It could easily deteriorate to the worst of all countries. It is most important that the Mathematical Association take some steps to "plug the hole". The hole could otherwise grow beyond repair.

S. INMAN

Analytic Geometry. By CLYDE E. LOVE and EARL D. RAINVILLE. 5th edition. Pp. xiv, 302. 28s. 1956. (New York and London : The Macmillan Co.)

Since there has been no review of this book in the *Gazette* since that of the first edition in 1923, it may be appropriate to say a few words about the fifth edition.

There is very little to distinguish this book from many other standard works on the subject. Both plane and solid analytical geometry are treated: in each case, no previous knowledge is assumed, except a course on elementary pure geometry.

In plane geometry, the authors deal with straight lines, circles, conics given by their simplest equations, and the general equation of the second degree. Other topics are algebraic curves of higher degree, and the graphs of certain transcendental functions.

In solid geometry, there is a fairly full treatment of planes and straight lines, and a very brief introduction to quadric surfaces.

The book is attractively printed, with the main results standing out in heavy type, and there are nearly 200 well-drawn diagrams.

E. J. F. PRIMROSE

Géométrie Euclidienne. By ROBERT BRISAC. Pp. 77. 1200 fr. 1955. (Gauthier-Villars, Paris)

If Euclidean geometry is to be studied in its own right, and not as a special case of projective geometry, there are two essentially different methods of approach (as there are for projective geometry itself). On the one hand, we may define our basic concepts (point, line, plane, incidence, order, parallelism, continuity, congruence, and so on) algebraically. The geometry then has no axioms in the ordinary sense, though we are, of course, assuming the axioms of algebra. On the other hand, we may regard our basic concepts as undefined except in so far as they satisfy a set of axioms which we lay down. This was essentially the method of Euclid, although, as Hilbert and Klein pointed out, Euclid tried to define everything, without much success, and omitted some necessary axioms.

The first axiomatic treatment of Euclidean geometry which satisfies modern standards of rigour was Hilbert's *Grundlagen der Geometrie*. Hilbert regarded as fundamental the idea of *congruence* of distance and angle. One can then introduce the idea of a *displacement*, or rigid motion, which transforms every distance and angle into a congruent distance and angle.

Early this century, Borel pointed out that it is possible to regard the idea of a displacement as fundamental, and then to define two distances (or angles) as congruent if they are connected by a displacement. The present book, by the late Professor Brisac, is an account of this method.

After the usual treatment of incidence of points, lines and planes, the idea of *partition* is introduced. Each line is regarded as being divided into two half-lines by any point of the line; half-planes and half-spaces are introduced in a similar way. This leads to a simple definition of order. Next comes the main concept, that of a displacement. All the other concepts, such as parallelism, continuity, and the measure of a distance or an angle, are treated in terms of displacements. The axioms therefore fall into five groups :

- (i) incidence,
- (ii) order (in a half-plane and a half-space),
- (iii) displacements,
- (iv) parallels (the usual Euclidean axiom),
- (v) continuity (expressed in terms of displacements, but equivalent to Hilbert's).

As Poincaré pointed out, in our early mathematical development, the idea of a displacement comes before that of distance and angle. In fact, one must displace a ruler (or protractor) in order to measure a distance (or angle). From this point of view, therefore, the present method has much to recommend it. In addition, the axioms are all quite simple and natural.

This is an important book, which should be read by anyone interested in the teaching of elementary geometry. For, although the treatment could not be presented in schools as it stands (except to the most able pupils), it gives the reader a very clear account of how Euclidean geometry can be set up with complete rigour while using the most primitive notions of space.

E. J. F. PRIMROSE

Übungsaufgaben zur höheren Mathematik. II. III. By R. ALBRECHT and H. HOCHMUTH. Pp. 131, 128. DM 9.80 each. 1955 (Oldenbourg, Munich)

A useful, if not very inspiring collection of worked examples on determinants, complex numbers, vectors, the calculus for more than one variable, surfaces and twisted curves, ordinary and partial differential equations. T. A. A. B.

Mathematical Analysis. By D. A. QUADLING. Pp. vi, 264. 25s. 1956.
(Oxford University Press)

Quadling's *Mathematical Analysis* is a very fine book, remarkable for its thoroughness, and its careful and lucid exposition; it is written with the loving attention to detail of a great German treatise.

The plan of the book is to postulate the existence of the least upper bound of a bounded set; of a root of the equation $f(x)=k$ for any k between two values of a continuous function; and of bounds for a function continuous in a closed interval. In an Appendix the three postulates are derived from Dedekind's Theorem (which is left unproved).

As an example of the scrupulous care the author takes, here is the statement of the theorem on composite limits (p. 39). *If x, y and z are variables such that z is a function of y which tends to c as y tends to b , and y is a function of x which tends to b as x tends to a ; and $y \neq b$ for any value of x in same region $0 < |x-a| < w$; then z tends to c as x tends to a .* The need for the condition $y \neq b$ is illustrated by a simple example.

The book is designed for the sixth form mathematical specialist and the first year University student; there may not be many schoolboys who will be able to read it but those who do will really catch a glimpse of the spirit of mathematics.

There are a number of small details which call for revision in a later edition. The proof (p. 46) that if $f(x)$ is continuous in (a, b) and takes each of its values once only then it is strictly monotonic, is not entirely clear. The first step should be to show that, since $f(x)$ takes each value once only, if $x_1 < x_2 < x_3$ are any three points in the closed interval (a, b) then $f(x_2)$ lies between $f(x_1)$ and $f(x_3)$; next, for any $x < X$ in the open interval (a, b) , if $f(a) < f(b)$ we deduce in turn that $f(a) < f(x) < f(b)$ and $f(x) < f(X) < f(b)$. The proof of 2.5 (p. 56) is incomplete; it was shown in 3.4 that if $f(x)$ is continuous in an interval then the inverse is continuous, but 2.5 requires the continuity of the inverse at $y=b$ when $f(x)$ is differentiable just at the single point $x=a$. The definition of infinite series (p. 85) appears to make no distinction between u_r and Σu_r ; i.e. between the sequence u_1, u_2, u_3, \dots and the sequence $u_1, u_1 + u_2, u_1 + u_2 + u_3, \dots$. The theorem on p. 104 should read: If there are numbers q, R such that, for all $r \geq q$, $|p_r/p_{r+1}| \geq R$, then $\Sigma p_r x^r$ is absolutely convergent for $|x| < R$. For the proof it suffices to observe that

$$\left| \frac{p_{r+1}x^{r+1}}{p_r x^r} \right| \leq \frac{|x|}{R} < 1$$

so that the result follows by D'Alembert's test (with $|x|/R$ itself as the " k " of the test). The use of a special function (p. 165) to show that a function with infinitely many discontinuities may be integrable seems rather pointless since it is just as easy to show that any function with discontinuities at a sequence of points like $1/n$, is integrable. The definition of differentiability (in 2 variables, p. 215) is inadequate to prove the formula for the derivative of $f(x(t), y(t))$ unless ξ_1 and ξ_2 in

$$f(a+h, b+k) - f(a, b) = Ph + Qk + \xi_1 h + \xi_2 k$$

are defined also at $h=k=0$; moreover in passing from this equation to the one with $k=0$, to prove P is in fact $f_x(a, b)$ we must replace ξ_1 by $\xi'_1 = \xi_1(h, 0)$, since ξ_1, ξ_2 are functions of h and k . On p. 230, in the proof that we may differentiate under the integration sign, the second derivative f_{xx} is introduced, whereas in fact the continuity of f_x suffices without making the proof any harder.

Many of the examples which carry the theory beyond the text are furnished with hints for solution ; two notable exceptions are Ex. 4, p. 193, on Newton's approximation (where a reference at least is urgently needed, and for section 5.1 should stand 4.1) and Ex. 8, p. 158, on the lower integral of the function $f(x)$ which takes the value $1/q$ when $x = p/q$, $(p, q) = 1$, and the value unity otherwise.

Of the many excellent features of the book which merit special reference, the chapter on integration is quite outstanding. The theory of the lower Riemann integral is carried further than is customary even in specialised treatises, and the account of substitution in the definite integral is seldom as correctly carried through. Another good illustration of the book's quality is to be found in the careful definition of a rearrangement and the proof that $\Sigma u_r = \Sigma u_{i(r)}$, when $i(r)$ is a rearrangement of the natural numbers (in a simple sequence). As a final illustration the account of Taylor's theorem (with Lagrange's remainder) deserves special mention for its genetic approach and the clarity with which it brings out the relation of the theorem to orders of contact.

R. L. GOODSTEIN

Differential Calculus. By W. L. FERRAR. Pp. x, 296. 27s. 6d. 1956. (Oxford University Press)

Ferrar's *Differential Calculus* is written more specifically for the University student. The second half of the book contains some excellent work on partial differentiation ; this part of the book is sound and reliable and treats such questions as Euler's theorem on homogeneous functions, and maxima and minima in several variable with care and insight. In the definition of the second differential (p. 196) it would be better to write

$$d^2f = d\left(\frac{\partial f}{\partial x}\right)dx + d\left(\frac{\partial f}{\partial y}\right)dy$$

rather than

$$d^2f = d(f_x)dx + d(f_y)dy$$

for in the case when $f = f(u(x, y), v(x, y))$ we require the operand of d to be $\frac{\partial}{\partial x}f(u, v)$ and not $f_x(u, v)$, which presumably means the result of substituting u, v for x, y in $f_x(x, y)$. Another pleasing part is the account of Rolle's theorem, but coming as it does from the pen of an author who has a so well-deserved high reputation as a writer of text-books, much of the book is very disappointing.

To begin with, the definitions, despite their preambles, are not well expressed. For instance, on p. 4, the definition of $\alpha_n \rightarrow 0$ stands as :

$$\epsilon > 0 ; \exists N . |x_n| < \epsilon \text{ when } n > N.$$

We are told that " $\epsilon > 0$ " means setting down any positive number ϵ whatsoever to begin with. " $\exists N .$ " means there is an N such that

Now the reader who knows all about convergence will be quite at home, but not the beginner. Do we just set down one value of ϵ and find N to prove convergence, or must we be able to find N (or N 's) for all values of ϵ . Is not N in fact a function of ϵ , and if so, why not say so? And what is meant by "*when* $n > N$ ". For one n , or for some n , or all n , or infinitely many n . These are precisely the points which must be made clear.

There are several errors, omissions and confusions, some perhaps of slight importance but some certainly serious. In the definition of differentiability if y is defined by the equation

$$k = f'(x)h + yh$$

then y is defined only for $h \neq 0$, yet the whole point of this form of the differentiability condition (and its application in the proof of differentiability of a composite function) lies in its validity for $h = 0$. This invalidates the proof on p. 33, since Δy may vanish without Δx vanishing.

It is apparently thought (p. 28) that it is an immediate consequence of $f'(x) > 0$ at all points of (a, b) that $f(x)$ is strictly increasing in (a, b) , yet in fact this result requires the mean-value theorem or some equivalent.

Chapter V contains a very brief sketch of the definition of real numbers and mentions their addition and subtraction (without proving anything); the reader is referred to Landau for details, but unfortunately Ferrar does not use Landau's definition of a real number but an older form. On p. 57 in this Chapter we meet the only attempt to prove a basic property of real numbers, that a bounded set has exact bounds. The proof meanders from "a little care is necessary to see that", to "but it is fairly clear from the definition". Who is to take the care if not the author? And if it is only *fairly* clear, why not make it quite clear—it is simple enough. The proof is so horribly inelegant for two reasons; one is the outmoded definition of real numbers, and the other is lack of a suitable notation.

The proof of Theorem 12 (p. 65) is false because $f(x)$ may change sign more than once in (a, b) ; we ought to consider the lower bound λ , not of values of x for which $f(x) > 0$, but of values of x for which $f(t) > 0$ for all t satisfying $x \leq t \leq b$.

It is by no means obvious on p. 67, in the proof of uniform continuity, that if X_1, X_2 are two points in (a, b) whose distance apart is less than the smallest interval of a finite family F of open intervals which covers (a, b) , then we can find overlapping intervals i_1, i_2 of F such that X_1 lies in i_1 and X_2 in i_2 . It is better to consider the class of end points of the intervals of F and choose X_1, X_2 so that their distance apart is less than any subinterval of (a, b) formed by any two end points of intervals of F .

On p. 100 it is stated that if $\sum \frac{h^n}{n!} f^n(a)$ is absolutely convergent for $|h| < R$, then the sum of the series is $f(a+h)$ provided that $|h| < R$. This is false as the example $f(x) = e^{-1/x^2}$, $x \neq 0$, $f(0) = 0$, shows. (This error is repeated on p. 103. It is also stated that at a singularity a single-valued complex function tends to infinity, but in the neighbourhood of an isolated essential singularity a function may approach any value we please.)

The chapter on extensions of the mean value theorem shows considerable technical weakness. The proof that

$$f^n(\xi) = \frac{\Delta^n h f(a)}{\Delta^n h g(a)} g^n(\xi)$$

occupies several pages but it is a trivial conclusion from the Extended Cauchy formula that if

$$F(a) = F'(a) = \dots = F^{n-1}(a) = 0; \quad G(a) = G'(a) = \dots = G^{n-1}(a) = 0, \quad G(a+h) \neq 0,$$

then there is a ξ in $(a, a+h)$ such that

$$F^n(\xi) = \frac{F(a+h)}{G(a+h)} G^n(\xi);$$

the deduction requires only the result

$$n^r - \binom{n}{1} (n-1)^r + \binom{n}{2} (n-2)^r \dots = 0, \quad r < n,$$

which is a particular instance of the formula $\Delta_h^n x^r = 0$ for $r < n$, which is assumed in Ferrar's proof. On p. 120, given for all h in a suitable interval,

$$f(a+h) = f(a) + hf'(a + \frac{1}{2}h)$$

we are asked to prove that if $f'''(x)$ is continuous then $f(x)$ is quadratic. In fact given only the existence of $f''(x)$ the result is trivial. Differentiating the given equation yields

$$f'(a+h) = f'(a + \frac{1}{2}h) + \frac{h}{2}f''(a + \frac{1}{2}h)$$

and taking $a = -\frac{1}{2}h$, it follows (writing $2x$ for h) that

$$f'(x) = f'(0) + xf''(0)$$

and so

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2f''(0).$$

R. L. GOODSTEIN

Famous Problems. KLEIN *et al.* 1955. (Chelsea, New York)

The Common Sense of the Exact Sciences. By W. K. CLIFFORD. \$1.60. 1955. (Dover, New York)

Famous Problems brings together Klein's *Famous problems of elementary geometry*, Sheppard's *From determinant to tensor*, Macmahon's *Introduction to combinatorial analysis* and Mordell's "Three lectures on Fermat's last theorem".

As the publishers say, there is no recondite reason for bringing these four works together—the reason is purely economic, since they would be too expensive reprinted separately. A very welcome companion volume to "Hobson *et al.*".

Clifford's famous work is very much in fashion at the moment and this reprint of the 1946 Knopf Edition is well timed.

R. L. G.

Calculus. By R. L. JEFFERY. Pp. xii, 242. 40s. 1955. (University of Toronto Press)

This Canadian text-book is written for students with no previous knowledge of the Calculus. Professor Jeffery remarks that first-year university courses in mathematics frequently presuppose a more complete background in algebra, geometry, trigonometry and statics than is assumed by available calculus texts and so a demand for streamlining has arisen, making the need urgent for a shorter road to the higher levels of mathematics.

In the first 76 pages, the reader learns how to find, when possible, the differentials and "anti-differentials" of every type of function that is likely to occur in his practical work. This high speed is achieved by laying down rules of which some are proved, others are left to the reader, and others are established in a chapter on "Fundamental Theorems" towards the end of the book, and by relying on a high degree of technical facility in handling mathematical processes; but it is possible to suspect that the goal will be reached at the cost of some casualties on the journey.

The book is designed to meet conditions which no longer apply to the curriculum in Great Britain. In 1905, *Ashford* and *Godfrey* tackled the problem of arranging the syllabuses at Dartmouth and Osborne so as to introduce naval cadets to the ideas and technique of the calculus at the age of 15; this led to the publication in 1910 of *Mercer's* admirable pioneer textbook on the calculus which in its turn, fortified by the increasing emphasis on science, encouraged the introduction of elementary calculus courses for specialist and non-specialist classes in secondary schools at a much earlier age than had been considered practicable; this is now common practice as is evidenced by the alternative syllabuses at ordinary level for the General Certificate of Education. Such courses rely on appeals to spatial geometry and are taken by students of whom only a very small proportion are interested in Analysis. This minority after a first elementary but comprehensive course proceed to some such book as *Hardy's Pure Mathematics* which contains (but in more detail and with many illustrations) the subject-matter of Professor Jeffery's Chapter XI on "Fundamental Theorems". It would be interesting to hear what proportion of those who read this streamlined first course are able and willing to digest this chapter; in some respects it would be easier and more instructive for them to tackle the first 44 pages of *Professor Jeffery's Functions of a Real Variable*.

In Chapter XI, the starting point is "the real number system, especially the decimal representation of real numbers". Professor Jeffery notes that "it is possible to go further back and study the structure of the real number system itself but that wherever the beginning, something must be taken for granted and long use has made us familiar with the real numbers". If this only means that the reader has used repeatedly such symbols as $\sqrt{2}$, it is undoubtedly true; but if it means that he has even a glimmering of an idea of how $\sqrt{2}$ can be regarded as an entity of a generalised number-system in which operations called addition and multiplication can be defined, it is improbable. Indeed it may be argued that the statements and proofs of fundamental theorems on bounds are meaningless unless prefaced by some account of the arithmetic of endless decimals. Some compromise must be made but it is open to serious doubt whether that chosen in Chapter XI is as suitable as either the brief introduction in *Professor Jeffery's Functions of a Real Variable* or the longer discussion in *Hardy's Pure Mathematics* or the still more detailed account in *Knopp's Infinite Series* or the Appendix of *Goodstein's Uniform Calculus*.

The course covers a wide range: applications to statics and hydrostatics, some differential geometry, infinite series, partial derivatives, multiple integration, three-dimensional geometry, first and second order differential equations, and vectors. It is built on a curriculum in use in freshmen classes at Queen's University for the past five years and may therefore be regarded as having passed the test of experience which is the only satisfactory criterion for a comprehensive book of this character.

C. V. DURELL

Altes und Neues über konvexe Körper. By H. HADWIGER. Pp. 102.
(Birkhäuser Verlag, Basle and Stuttgart)

This readable little book restricts itself entirely to the theory of convex bodies in space of at most three dimensions. Most of the results and techniques can be generalised to higher space, but the author, by being less ambitious in this respect, saves much in space and complexity. The book is concerned with the theory of convex bodies for its own sake, not with applications.

The subject matter is beautifully organised. The foundations, including Blaschke's selection theorem, Steiner's method of symmetrisation, and Minkowski addition, are laid in the first two chapters. Certain "intuitive properties", including the existence of tangent planes (*Stützebene*) are stated without proof. Though Steiner's formula for the volume of a parallel body is proved, no mention is made of mixed volumes in general.

The third and fifth chapters are closely related. In the third chapter it is shown that any functional $f(K)$ defined for all convex bodies and satisfying certain simple axioms differs by a constant from a linear combination of three fundamental magnitudes V (the volume), F (the surface area), M (the integral of the mean curvature). This remarkable result (together with other similar ones) enables the author to give extremely simple proofs of Cauchy's projection formulae (in Chapter III) and of various formulae in integral geometry (in Chapter V). It is clear that many more applications are possible.

The subject of Inequalities, treated in Chapter IV, is the oldest and most intensively studied aspect of the theory, yet it is still incomplete. The main problem, not yet solved, is to find criteria, in the form of inequalities, for a convex body to exist with given values of V , F , M . This chapter justifies the book's title, for it contains the classical results of Brunn and Minkowski side by side with recent work of Bol, Fejes Toth, the author and others. Most stimulating is the section on the unsolved problem just mentioned. The problem is transformed so that it is seen to depend on specifying a plane region in "Blaschke's diagram". The author tells us a lot about the region, though he is not able to specify it exactly. No mathematician can read this section without being moved to try the problem for himself. (All the reviewer's attempts have, alas, been unsuccessful!)

One minor criticism involves the choice of notation. The reviewer prefers the symbol \cup , instead of the addition sign $+$, to denote union of sets. This enables the $+$ sign to be kept for Minkowski addition, as in Minkowski's own work. This notation is more in keeping with modern usage in algebra.

The book is elegantly and economically arranged, and a surprisingly large subject-matter is covered, without painful brevity or undue condensation, in the slim volume of 100 pages.

A. M. MACBEATH

Les Épreuves sur Échantillon. By MAURICE DUMAS. Pp. 170. 1000 fr. 1955. (Centre National de la Recherche Scientifique, Paris)

This monograph on statistical tests is one of a series on applied mathematics now being published by the Centre d'Études Mathématiques en vue des Applications. It is written for industrial technicians and is intended to acquaint them with the results of academic research in the hope of getting speedier application of the results in industry.

The monograph is a concise exposition of the techniques which can be applied to the quality control of mass-produced manufactures. Great care has been taken to make the monograph intelligible to those who fear that statistics contains "trop de mathématiques pour leur goût"! To avoid mathematics and to keep the handbook within a practicable size, the author has omitted the derivations of the statistical formulae he quotes but demonstrates their application by using simple numerical illustrations. And so, in spite of an introductory chapter, in spite of the protestations and care of the author and of the authority of its sponsors, this monograph becomes simply a rather bewildering book of recipes.

The author's position is difficult. A fuller exposition would inhibit his intended readers, yet the application of quality control techniques in industry is an urgent need. Until the elements of statistics and probability theory are more generally taught all who find themselves in the same position as M. Dumas will have to adopt his compromise.

B. C. BROOKES.

Eingefangenes Unendlich. By F. VON KRBEK. 2nd ed. Pp. iv, 332. 1954. (Akadem. Verlagsgesellschaft Geest & Portig, Leipzig)

How shall we translate the title of this history of mathematics: "The Infinite Caught", or "Infinity Confined"? Two chapter headings are: "A jump in the alphabet: the Greeks" and "More is less". The author seems to follow a fashion for facetiousness and right through the book he cannot refrain from witticisms of a sort that one can almost hear his students guffaw, though older men may wince. Like some modern poetry the text appears to contain hidden allusions and unacknowledged quotations, and there are some jarring innovations in the syntax not otherwise encountered. In the actual history there are errors of detail almost everywhere. Yet this is an excellent book. The aim is not a chronological collection of important or amusing facts, but a selective study of the developments of some major mathematical ideas through historically documented stages. There is first an introductory chapter of "romantic" history, with life stories and anecdotes, grouped by subjects (e.g. Gauss, Abel, Galois for modern algebra) and beginning with authors not too distant in time. The chapter "Pythagoras would be amazed" traces the history of the concept of number from Egypt and Babylon to Dedekind sections and infinite sets, including the early calculus. In these 120 pages the author finds room for a proof of the fundamental theorem of arithmetic and for a concise account of the theory of groups. Altogether there is a good deal of real mathematics in the book: thus when in a typical example of tracing a development we meet figurate numbers in Greek mathematics this idea is followed through to lattices and unimodular substitutions, and not only in words but with figures and formulae. The second main section of the book, "Euclid surpassed", deals with geometry from the pyramids to the manifolds of topology, and although I do not believe that the general reader can follow it, the development shows just how and why Euclid was surpassed, how in fact by generalisation and deeper abstraction the old becomes embedded in the new without losing its own relevance, and how unification is often the result of re-thinking. It is this effort of re-thinking, of translating into one's own language, which makes historical studies valuable for the mathematician. The author does not shirk the challenge of a true problem, whether it is a better interpretation of a difficulty with Egyptian fractions, or "the enigma of genius", or again a remark on the difference of Greek and Indian mathematics and their modern synthesis. It is indeed a valuable work, and as for some lapses, we are warned that "who does not at some time correct history, is but a boring pedant". A. PRAG

La Géométrie et le Problème de l'Espace. By FERDINAND GONSETH. Vol. VI. **Le problème de l'espace.** Editions du Griffon, Neuchâtel, Suisse. 11.70 Swiss francs. 1955.

Space and the problems connected with space loom threateningly in the consciousness of modern man. Every day the popular press and the radio pump out an ever-increasing ration of discussion on space travel, guided missiles and the possibility of supermen on flying saucers having already landed on earth. Who has not heard of Jet Morgan? Whoever he is, such ignorance will immediately disqualify him for a job as a leader of modern youth! The atmosphere around us is so thick with willing credulity that not even an Astronomer Royal can shake the public belief that men will land on the moon in our lifetime.

We have, of course, a good deal of information about the properties of space near the earth's surface, and even more confidence that space beyond cannot be very different. But in fact, leaving faith, hypothesis and conjecture on

one side, we know very little about outer space. Our main source of information is simply *light*. Now light may very well behave in an unforeseen manner in outer space.

To illustrate this possibility, let us consider the behaviour of a beam of radio waves. At one time such a beam was thought to describe a straight line, irrespective of the distance of the wave front from the earth. If this conjecture had indeed been true, Marconi would not have received in Nova Scotia a radio signal emitted in England. In fact a radio beam does not describe a straight line, but is deflected downwards by an ionised layer in the upper atmosphere, now called the Heaviside layer. In other words, the geometry of space near the earth's surface is not euclidean if our fundamental construct, called a line, is a radio beam.

The path of a beam of light is fundamental to any discussion of the geometry of outer space. If we look along it, we must perforce regard it as "straight"; that is, there is no alternative but to regard light-beams as describing geodesics. But for all we know, light-beams may whirl in dizzy spirals before describing straight lines when they approach the earth's surface. From the earth itself we have no means of checking on such undignified behaviour. On the other hand, if the artificial satellite soon to be launched by United States scientists does succeed in penetrating outer space, we shall obtain information, from observations made in outer space, which cannot be obtained, by any process of induction, on the earth's surface. But the artificial satellite may well be deflected back to earth by a new kind of layer, for all that anyone knows. Then we shall have advanced in knowledge, but in an unexpected direction.

All this may be thought to be idle speculation, but nowadays such speculation is to be regarded as scientific if there is even the flimsiest evidence to support it! At least, a study of recent scientific history would lead an impartial reader to such a conclusion. Who has not heard of the expanding universe? Great scientific reputations have been erected on the assumption that distant nebulae are retreating at ever-increasing speeds from the benighted solar system. Weekly radio talks on the continuous creation of matter have thrilled millions. If the universe is expanding, matter must be created to keep pace. Once cosmologists start off, they have unbounded space in which they may romp with their fancies. But what started the expanding universe idea (for we must not talk about it as a fact, as too many do)? It was begun by the observation that in the spectrum of the light from distant nebulae, there is a minute shift in the red line. It is doubtful whether the consequent hypothesis of an expanding universe is the simplest which might follow the observation of a shift in the red line. But cosmologists are the poets of modern times. Shelley would certainly have been a cosmologist, had he been born in the twentieth century.

These uncoordinated thoughts on space and our knowledge of space have been occasioned by reading the final volume of Prof. Gonseth's *magnum opus* on geometry and the problem of space; but Prof. Gonseth is in no way responsible for what is printed above. His work, which does indeed include a discussion as to whether measurement can show whether space is euclidean or non-euclidean, is a refreshing survey of some of the geometries at present known, together with a study of the various categories of axioms on which a geometry can be based. An interesting section of the book deals with the attitudes towards geometry of various great geometers: Clairaut, Legendre, Gauss, Bolyai, Lobatschefski, Pasch.

The one exception which might be taken to anything in this admirable work is to the assertion that our experience of space inevitably leads to certain axioms about points, lines and planes. Anyone who has ever tried to teach elementary three-dimensional geometry to a class of extroverts might well

question this assertion. Humans seem to differ from each other in their fundamental concepts of space. Much must depend on one's fundamental physiological apparatus of sight, hearing and touch. Experimental psychologists would confirm this.

One's impression of objects on the right of one is inevitably different from the impression of objects on the left. Objects which lie above the normal level of sight are different from objects which lie below. The full moon rising over a perfectly flat landscape looks far larger than when it is directly overhead. Tell a company of intelligent, non-scientific people, as I once did, that the moon is in fact of the same apparent size in each case, subtending the same angle at the eye, and observe their attitudes of indignant denial. Space-perception varies from person to person, and space can hardly be said to be homogeneous, even for one given person.

The problems still to be solved in space, even at a very elementary level, are therefore manifold, but they are not purely mathematical ones. Nor are they purely philosophical ones either. The experimental psychologist will need to be both mathematician and philosopher if he is to tell us what our perception of space really is.

D. PEDOE

The third dimension in chemistry. By A. F. WELLS. Pp. x, 143 + 16 plates. 21s. 1955. (Oxford University Press)

This volume by Mr. A. F. Wells gives a new approach to the early study of crystallography as viewed by the chemist. The first four chapters deal with the geometry of crystal structure and the two remaining chapters with the main types of crystal to be met with in chemical compounds. In the preface, the author states that he is concerned with "the problem of mental outlook" in so far as structural chemistry is concerned and he leads the reader through a survey of "polygons and plane nets", "polyhedra", "repeating patterns" to the "shape and symmetry of crystals" and thence to a consideration of the application of these structures to chemical entities.

It is, however, a little difficult to decide for whom this book is intended, since the introduction would suggest a reader with little knowledge of chemistry. On the other hand, in the later chapters, the descriptions of atomic structure, chemical bonds, and their correlation with the Periodic Table are too briefly dealt with for the non-chemist, whilst the chemistry student should be thoroughly conversant with the ideas expounded therein. None the less, this volume provides a refreshing viewpoint for the chemist in its approach from mathematical considerations and gives a valuable introduction to the author's earlier and more extensive work *Structural Inorganic Chemistry*. The stereoscopic diagrams and photographs are a most interesting feature of the numerous illustrations.

F. R. SHAW

Advanced Level Applied Mathematics. By S. L. GREEN. 12s. 6d. (with answers). (University Tutorial Press)

This excellent text-book is a modern version of the author's *Intermediate Dynamics and Statics*, and provides a suitable course in Mechanics and Hydrostatics for candidates at the Advanced or Scholarship levels, for non-specialists.

The book is written in a formal style, and the matter is set out deductively in a manner readily understandable by the average student, with a wealth of worked examples. A beginner at the subject, who had a working knowledge of the calculus, could not do better than work at this book. The full sets of

examples at the end of each chapter, graded up to "Advanced Level" standard, are uniformly good. The book does not cover the requirements of either mathematical specialists or candidates for University awards in science, but provides an admirable "first course" for such students.

A number of criticisms of individual points follows.

In the discussion on Newton's laws, it is not made clear how the concept of mass is developed—mass is regarded as a prior concept, needing no explanation.

The usual analytical conditions for equilibrium of a rigid body are given very little emphasis, the main emphasis being on three-force problems, and graphical or semi-graphical methods. The only case where the equilibrium of a framework of heavy rods is considered is at the end of Chapter 20, near the end of the book. More examples on this work are needed. The conditions for equilibrium are nowhere proved to be both necessary and sufficient for equilibrium.

In the work on friction, the possibility of reactions being indeterminate is not mentioned.

Work on the principle of energy, stability of equilibrium (the energy test) and rotation of a rigid body about a fixed axis, is all done in one chapter, with a set of only 13 examples at the end of the chapter. This is clearly not enough to provide adequate practice for the reader, and denotes an insufficient attention to rigid dynamics on the part of the author.

In the copy sent for review, the paper used was decidedly "off-white" in colour, and the ink impression rather faint, but the layout is very attractive, and neat. The examples are printed in rather small type, and the thinness and transparency of the paper are a little uncomfortable for easy reading and handling.

F. J. TONGUE

Allgemeine Theorie der algebraischen Zahlen. By PH. FURTWÄNGLER, H HASSE and W. JEHNE. Enzyklopädie der mathematischen Wissenschaften. Band I/2. Teil : C. Reine Zahlentheorie. Heft 8, II, Artikel 19. Pp. 50. 1953. (Teubner, Leipzig)

The Enzyklopädie der mathematischen Wissenschaften aims, in its articles, at giving a brief but reliable account of results and literature for the specialist, as well as an introduction and general view for the beginner who has a basic knowledge of mathematics. The present article on the General Theory of Algebraic Numbers fulfils this aim admirably on a space of only 50 pages. It introduces the reader not only to the classical theory, as can be found e.g. in Hecke's Theorie der algebraischen Zahlen, but also to the modern approach by means of local theories (p -adic numbers), as given e.g. in Hasse's Zahlentheorie. The theory of special fields, class field theory, and the general reciprocity laws are, however, excluded as they will be treated elsewhere in the Enzyklopädie.

After giving a list of text books and monographs, the article deals successively with the following items : History. Algebraic integers, and divisibility. Dedekind's ideal theory. Kronecker's method of indeterminates. The divisor-theoretical basis of arithmetic according to Kummer and Hensel. The structure of the residue class-ring modulo an integral ideal. Ideal classes. Differents and discriminants. The decomposition of prime ideals in extension fields. Normal extensions. Units. Decomposable forms and Klein's geometrical interpretation. The composition of number fields. The theory allied to Artin's L -functions. The analytic determination of the class number.

K. MAHLER

Stochastic Models for Learning. By ROBERT R. BUSH and FREDERICK MOSTELLER. Pp. xvi, 365. 72s. 1955. (John Wiley and Sons, New York; Chapman and Hall, London)

The jacket of this book, one of the Wiley Publications in Statistics, claims that it is the first extensive attempt to present a probabilistic analysis of data obtained in learning experiments. Learning here means "a systematic change of behaviour, whether or not the change is adaptive, desirable for certain purposes, or in accordance with any other such criteria. Learning is complete when certain kinds of stability, not necessarily stereotypy, obtain". Behaviour is, in the view of the writers, essentially probabilistic, i.e. it is measured by the probability of occurrence of a given class of responses. Whenever an experimenter manipulates a subject's environment in a specified way an *event* has occurred, and a *trial* is an opportunity of choosing among a set of mutually exclusive and exhaustive alternatives or *responses*. The occurrences of events correspond to mathematical operators which change the probabilities of alternative responses. The writers are interested in the cumulative effect of a sequence of events. Learning is represented by orderly changes resulting from the occurrences of events, events being such things as stimulus change or actual response occurrences.

The writers have conducted a research that in many ways overlaps with either the "reinforcement theory" (e.g. reduction of basic drives lead to learning) associated with the names of Thorndike, Hull and Spence, or with the "contiguity theory" (e.g. conditioning is a consequence of the contiguity between stimulus and response) associated with the name of E. R. Guthrie. Their development is on the following lines.

Let us consider the case of two mutually exclusive and exhaustive alternatives A_1 and A_2 . Let p and q be the probabilities of A_1 and of A_2 respectively (say, A_1 is a "success" and A_2 is a "failure"). After the trial, suppose that the new probabilities are respectively $u_{11}p + u_{12}q$ and $u_{21}p + u_{22}q$. Then it follows from the probability invariance rule that $u_{11} + u_{21} = 1$ and $u_{21} + u_{22} = 1$. Put $u_{12} = a$ and $u_{21} = b$. Let $\alpha = 1 - a - b$ and $\lambda = a/(1 - \alpha)$. Then, writing Q as an operator, we have the equation $Qp = a + \alpha p = \alpha p + (1 - \alpha)\lambda$. The object of the book is to investigate the utility of this equation and of its generalisations for describing detailed performances or sequences of responses observed in various experiments in the study of learning.

The writers are two professors in the Department of Social Relations in the University of Harvard, and their treatment is the outcome of seminars, conferences, and so forth over the past six years with a number of persons in the States interested in the problems considered. They have written from time to time in the interval in American journals on the problems now dealt with. Data on animal and human learning present peculiar problems to the statistician. As irreversible changes take place while the data are being collected, repeated sampling is seldom possible. Organisms that can be considered identical at the start of an experiment do not remain completely identical, because each has a different history during the course of the experiment. The writers apply their work to such experiments as maze running by rats, learning of nonsense syllables by subjects, and "Monte Carlo" experiments (by what they call "stat-rats") using tables of random sampling numbers to decide the outcome of various alternatives within an artificially built series. They suggest procedures by which estimates of the three parameters can be made from the data, e.g. by using a maximum likelihood process. Tables for six functions that they need for estimates are given at the end of the book. They also suggest methods of testing the extent to which their estimates agree with the data. They reject the usual "goodness of fit" test in favour of a run test.

The writers claim that the three parameters that they use (*a*) give a detailed

description of the data, (b) lead to a concise summary of the data and (c) provide a base line for studying effects outside the model. They suggest that the "natural" parameters p , a' , and b are less useful in this way than are their three, of p , the probability of a response, α , the ineffectiveness of the event, and λ , an element of the limit vector (if there is perfect learning then $\lambda=1$). They point out some important defects of their inquiry. Among these are (a) the assumption of "path independence" (i.e. the assumption that a particular outcome following a given response changes the set of r probabilities in a unique way that is independent of earlier events in the process—thus omitting effects of memory, practice effects, or long range effects of trauma), (b) linearity of the transforming vector operator, (c) failure to handle response intensity, (d) failure to deal with discrimination, response chaining, etc.

In the majority of cases their fundamental equations are not easily soluble, if at all, but they consider certain special cases. For some they assign special values to parameters. Such are the case of the "identity operator Q' " (which does not change p), the "equal alpha" condition for two responses A_1 and A_2 and two events E_1 and E_2 (E_1 has the same effect on response A_1 as E_2 has on A_2), and the case when success eventually occurs on all trials (the case of "perfect" learning). They consider such kinds of experiments as those that they call "experimenter-controlled", "subject-controlled" and "experimenter-subject-controlled".

The writers suggest that the possible generality of the mathematical system may suggest to some readers quite different applications from those that they discuss. To a reader who does not look on learning from such a definitely behaviouristic point of view as do the writers, this appears to be a legitimate hope.

FRANK SANDON

Aperçu de la Théorie des Polygones Réguliers. By PIERRE A. L. ANSPACH. Pp. 92. No price. Privately printed by the Author. 94 Rue Berckmans, St. Gilles, Bruxelles.

The title of this monograph is somewhat misleading, since the only polygons treated are those with 7, 11 and 13 sides. The author's main concern is with the "heptal" triangle, with sides $2 \sin \pi/7$, $2 \sin 2\pi/7$, and $2 \sin 3\pi/7$, whose sides are equal to those of the three regular heptagons (convex and stellated) that can be inscribed in a circle of unit radius. A large number of concurrences, similarities and so forth are discovered among points associated with the triangle, and certain segments are shown to have rational or quadratic irrational lengths. Typical results that are simple enough to be briefly stated are (i) the nine-points centre F of the heptal triangle and its reflexion in a certain diameter of the circumcircle are the Brocard points of the triangle, and the sine of the Brocard angle is $1/8^4$; (ii) F is the centre of a lemniscate of Bernoulli which passes through the vertices of the triangle, and whose foci are the centre of the smallest ex-circle of the heptal triangle and another vertex (10) of the 14-gon associated with it; (iii) the pedal triangle of the heptal triangle is similar to it, with cyclic permutation of the sides. The methods used are wholly elementary and large use is made of barycentric coordinates. Several features make the work very difficult to follow. For the most part, results only are given, and the reader is referred to duplicated memoirs for details. Vertices are numbered in the scale of 5, those of the regular 14-gon being Z (for zero?), 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22 and 23. The heptal triangle is Z , 2, 11. The reviewer can find no reason for this, which is very confusing. All trigonometrical results are given in terms of

lengths of chords rather than trigonometrical functions of angles. Square roots are either abbreviated to single letters or written out in words. Points are christened with strange names, Evenit, Bab, Lol, Rid . . . for which no explanation is forthcoming. The author writes highly colourful French and describes his results with boundless Gallic enthusiasm, which he has evidently found difficulty in inducing others to share. He is like a man who has gone into a strictly delimited wood and is overjoyed to find there five trees in line or three at the corners of an equilateral triangle. The wood as a whole has significance in the mathematical landscape, but most of us will feel that the individual trees have not. The Victorian collector of curiosities is out of fashion.

H. M. CUNDY

Humidity. By H. L. PENMAN. Pp. 71. 5s. 1955. (Institute of Physics)

This monograph is one of a series intended for general reading by students in courses for the Higher National Certificate in Applied Physics. It consists mainly of descriptions of the various forms of humidity measuring devices used in industry and meteorology. There are also chapters on the physical principles of vapour-liquid equilibrium and the importance and applications of industrial hygrometry.

Throughout the monograph, numerical examples are given to emphasise the orders of magnitude of the quantities involved; the descriptions are clear, concise and well illustrated. The result is that most of the chapters are very "readable" and fulfil their purpose.

The first two chapters, on the theoretical principles however, seem to have been over-simplified and do not carry the same conviction. Some of the basic concepts such as partial pressures and osmosis are hardly explained at all.

The well-known hydrostatic method of deducing the equilibrium equations for a curved surface and a solution is used. The artificiality of the model is pointed out by the author, but a simplified thermo-dynamical argument would have overcome the difficulty without exceeding the standard at which the monograph is written.

C. A. HAYWOOD

Théorème sur les Surfaces d'onde en Optique Géométrique. By RENÉ DAMIEN. Pp. 34. 1955. (Gauthier-Villars, Paris)

In this book the author applies a theorem which is a direct consequence of the law of Malus and Dupin to special cases of reflection and refraction in simple optical systems. The basic theorem is: "If h is the wave surface of light from a point source S after refraction at an interfacial surface g between two media, then a surface g' , inverse to g with respect to S will be the wave surface of light from S refracted by a surface h' , where h' is inverse to h ."

Application of the theorem affords a very concise and elegant method for the determination of wave fronts. Results, many of which are familiar, are obtained for reflection and refraction of light from a point source at spherical and other associated surfaces.

The method may be extended to deal with line sources, but offers no advantages over direct methods in more general cases. Thus the problem of multi-component optical instrument design is beyond the scope of this treatment.

The book concludes with a description of a mirror system (*miroir intégral*) in which all the light from a point source is reflected into a cone, the rays all radiating from the apex.

C. A. HAYWOOD

Monographs on Topics of Modern Mathematics. Edited by J. W. A. YOUNG. Pp. xvi, 416. \$1.90 (paper-bound). 1955. (Dover Publications, Inc.)

The Dover Co. has laid all students who are specialising in mathematics under an obligation by producing this reprint. First published in 1911, it retains its freshness, and could be a valuable adjunct to the reading of a first-year University student by introducing him to results which are normally accessible only in advanced treatises, by widening his horizons and perhaps crystallising his choice of specialist study. Modern taste has dictated some slight changes in the terminology of "Non-Euclidean Geometry" (F. S. Woods), and there are now more fashionable approaches to the complex number system than that contained in Huntington's "Fundamental Propositions of Algebra"—an excellent article this. Other topics are "The Foundations of Geometry" (O. Veblen), "Modern Pure Geometry" (T. F. Holgate), "The Algebraic Equation" (G. A. Miller), "The Function Concept and the Fundamental Notions of the Calculus" (G. A. Bliss)—this is now largely to be found in school text-books—"Theory of Numbers" (J. W. A. Young), "Ruler and Compass Constructions, Regular Polygons" (L. E. Dickson), and "The History and Transcendence of π " (D. E. Smith). An introduction by Professor Morris Kline does much to bring the bibliography up to date. The few misprints will not worry the student who can read this book intelligently. It can be strongly recommended to the pure mathematician, and is cheap at the price; a cloth-bound edition costs \$3.95.

B. A. SWINDE

Cours de Géométrie Infinitésimale. Fascicule V, Géométrie Infinitésimale. Deuxième Partie : Théorie des Surfaces. By GASTON JULIA. Second Edition. Pp. 145. 2400 fr. 1955. (Gauthier-Villars, Paris)

This book is a continuation of fascicule III which gave an account of the theory of curves. It contains four chapters—chapter XV deals with general properties of surfaces and of curves traced on surfaces; chapter XVI with particular curves on surfaces, lines of curvature, asymptotic lines, conjugate nets; chapter XVII with congruences of lines, and the last chapter XVIII with applicable surfaces and conformal representation. The fascicule can be read with little effort since the description and arguments are lucid and the printing is excellent.

With the appearance of this fifth and final fascicule of the second edition of Julia's "Cours de Géométrie de l'École Polytechnique", it may be useful to look back at the parts previously published and to consider the work as a whole. The author set out to produce a work on differential geometry which would be based on a firm analytical foundation so that the treatment could be both rigorous and lucid. There is no doubt that he has achieved his object. In fact, if one wished to recommend a text-book which would serve as an introduction to Darboux's 4-volumed "Théorie des surfaces" (Gauthier-Villars, c. 1887), the present work would be most suitable. Its style is that of the classical French "Cours d'Analyse", and it is not without significance that references are nearly all to works of Darboux and Goursat.

It may seem unfair to criticise a book because it does not do (and makes no pretence at doing) what the reviewer hoped it would do. Even so, it is disappointing to feel that in spirit this work really belongs to the last century. Certainly the work gives no indication of ideas and methods which have played an important part in the development of differential geometry during the past thirty years. For example, a surface is defined by its parametric equations referred to a set of Cartesian axes in 3-dimensional Euclidean space. The problem of defining a surface intrinsically is not mentioned, and topological questions are completely ignored.

If, instead of writing what is essentially a 5-volumed introduction to Darboux's "Théorie des surfaces", the author had used his considerable powers of exposition to give a similar introduction to Élie Cartan's "Leçons sur la Géométrie des Espaces de Riemann", the resulting course would have been equally valuable as an intellectual discipline and, in addition, it would have given the student at least some idea of topics which receive the attention of differential geometers today.

T. J. WILLMORE

Colloque sur l'Analyse Statistique. Pp. 186. 1900 f. 1955. (Centre Belges de Recherches Mathématiques : Thone, Liège ; Masson, Paris)

This book contains the papers (without discussion) read at the colloquium held in Brussels in 1954. Of the ten papers three discuss aspects of stochastic processes (they are by M. S. Bartlett, A. Blanc-Lapierre and D. Dugué) and two discuss topics arising from the theory of games (they are by P. Gillis and S. Huyberechts and by E. Franckx). The five remaining papers on miscellaneous topics include an elegant contribution to regression theory (by G. Darmois) and a discussion of the fundamental problems of statistical inference (by B. de Finetti).

The authors report on their recent achievements and discuss their unsolved problems. The book is therefore likely to interest the research statistician rather than the general reader.

B. C. BROOKES

Reelle projektive Geometrie der Ebene. By H. S. M. COXETER. Translated from the 2nd English edition by W. Burau. Pp. 190. DM 18.60. 1955. (Oldenbourg, Munich)

Readers of the *Gazette*, to whom Professor Coxeter's book has already been strongly commended, will no doubt prefer to have the English original. But this translation is a well-deserved compliment to a valuable work, and a tribute to the merits of the now flourishing Cambridge school of geometry. The printing is pleasant and the diagrams are admirably clear.

T. A. A. B.

THE MATHEMATICAL ASSOCIATION

Intending members of the Mathematical Association are requested to communicate with one of the Secretaries, Mr. F. W. KELLAWAY, Miss W. COOKE. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of *The Mathematical Gazette* and a copy of each new report as it is issued.

Change of Address should be notified to the Membership Secretary, Mr. M. A. PORTER. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The address of the Association and of the Hon. Treasurer and Secretaries is Gordon House, 29 Gordon Square, London, W.C.1.

December, 1956

REPORT OF THE COUNCIL FOR THE YEAR 1955

Membership.

During the year ended 31st October, 1955, 182 ordinary members and 54 junior members were admitted to the Association. At the end of the year the membership figures were : Honorary, 7 ; Ordinary, 2,357 ; Junior, 93 ; Life, 242 ; a total of 2,699 compared with 2,592 at the beginning of the year.

It is with regret that the Council reports the death of the following members : Mr. G. W. Brewster (1911), Mr. F. Burgess (1946), Dr. H. S. Carslaw (1903), Lord Charnwood (1942), Miss I. M. Giles (1955), Prof. P. J. Heawood (1892), Mr. J. A. Holden (1922), Mr. S. G. Horsley (1934), Miss D. L. King (1937), Miss F. M. Pickup (1939), Mr. A. S. Ramsey (1905), Mr. G. I. Stratton (1909,) Mr. H. H. Thorne (1920), Mr. F. Underwood (1928), Mr. F. J. Wood (1947) ; and also of an Honorary Member, Professor R. C. Archibald, the eminent authority on the history of mathematics, and of Dr. H. R. Hassé (1920), who was President of the Association in 1950.

Plans are in hand for the publication during 1956 of a printed Membership List.

Finance.

The accounts this year show an excess of income over expenditure of £189 17s. 4d., almost exactly^o the same result as last year. The amount available for reports and library binding was estimated to be £1,000, and the expenditure of this sum would have resulted in a small decrease in the general fund. In fact only £778 was spent on these two items ; the remainder will make a useful contribution towards the requirements of next year.

The amounts received from subscriptions and from the sales of reports were in both cases more than £100 above last year's figures. It seems possible that they will show a further increase next year. With printing costs about to rise again, our general position will deteriorate unless the income from subscriptions also rises. It remains true that the principal factor in balancing our accounts is a close relation between income from subscriptions and the cost of producing and distributing the *Gazette*.

The War Loan stock remains intact at £1,100 and the cash in hand is now £630. This is a small enough reserve when the estimated expenditure for 1956 is over £4,500. Fortunately the bulk of our annual income arrives in the first half of the year and so we should manage to avoid an overdraft. The prospect for 1956 is fair ; it should be possible to provide about £1,100 for printing and reprinting reports, with another £100 for library binding. If these amounts are actually spent there may be a drop of about £500 in the general fund.

The Mathematical Gazette.

Professor T. A. A. Broadbent retired from the Editorship after holding the office for twenty-five years and in January 1956 an Editorial Board was instituted. Under Professor Broadbent's direction the *Gazette* grew to be a Journal with an international reputation, its standing abroad being evidenced by the immense volume of periodicals which the Association receives in exchange for the *Gazette*. Reviews in the *Gazette* are quoted in Mathematical Journals in such diverse countries as Russia and America and testimony to the *Gazette*'s value to teachers and research workers alike comes in almost daily from all over the world.

The new Editorial Board will continue the policy of encouraging authors to produce articles of the widest possible mathematical interest. Good elementary expositions of branches of modern mathematics will be especially welcome.

Lack of space will continue to oblige the Board to decline articles of perhaps considerable intrinsic merit if their appeal is too limited.

Library.

During the session 1954-55 there has been a substantial increase in the number of books and periodicals issued; the number of volumes borrowed amounted to 174, of which 129 were books and 45 periodicals. This increase is probably due to the publicity given to the Library during the Annual General Meeting in Leicester, in April 1955.

Binding is continuing and a start has been made on binding periodicals with only comparatively short runs.

Accessions to the Library continue to be confined to periodicals received in exchange for the *Gazette* and to generous gifts.

The Teaching Committee.

The Teaching Committee met at Leicester on April 13th, 1955. Miss Mary Hartley was elected to fill the vacancy caused by the departure of Miss Christine Hamill to Ibadan.

Since that meeting the *First Report on the Teaching of Geometry* has been reprinted. A reprint of the *Report on the Teaching of Algebra in Schools* has been called for, but a slight revision is first being undertaken by correspondence; § 13 has been re-written and about 40 minor alterations recommended. The Committee is happily favoured to number the Chairman of the original Algebra committee (Mr. C. O. Tuckey) and three other members among its present members.

In preference to calling a meeting of the whole Teaching Committee during the year, the Chairman circulated to all members at the beginning of October an interim report on the work of the various sub-committees, and a statement (which was also made available at a Council meeting) showing the annual sales of the Association's Reports since 1945.

The *Report on the Teaching of Mathematics in Primary Schools* was published in November. A complimentary copy was subsequently sent to every Chief Education Officer and Director of Education, and to Heads of Training Colleges, with a covering letter from the Chairman explaining briefly the aims of the Association and commending the Report to the appropriate member of staff. Copies will also be sent to suitable journals for review.

The Modern Schools sub-committee has been strenuously active. The one reporting on Sixth Form Algebra has suffered various hindrances. The one dealing with Sixth Form Analysis has completed a brief report on Course I—intended to cover the needs of science pupils taking mathematics. It is hoped that the draft may soon receive the approval of the Teaching Committee, the Council and the Editor for being printed and issued as an inset to the *Gazette*.

A four-page leaflet giving particulars of the Association and of its publications has been printed and copies are now inserted in all Reports other than those issued on publication to members.

The Branches.

The work of the Branches has been continued with many interesting meetings. Some Branches have also arranged problem-drives or quizzes.

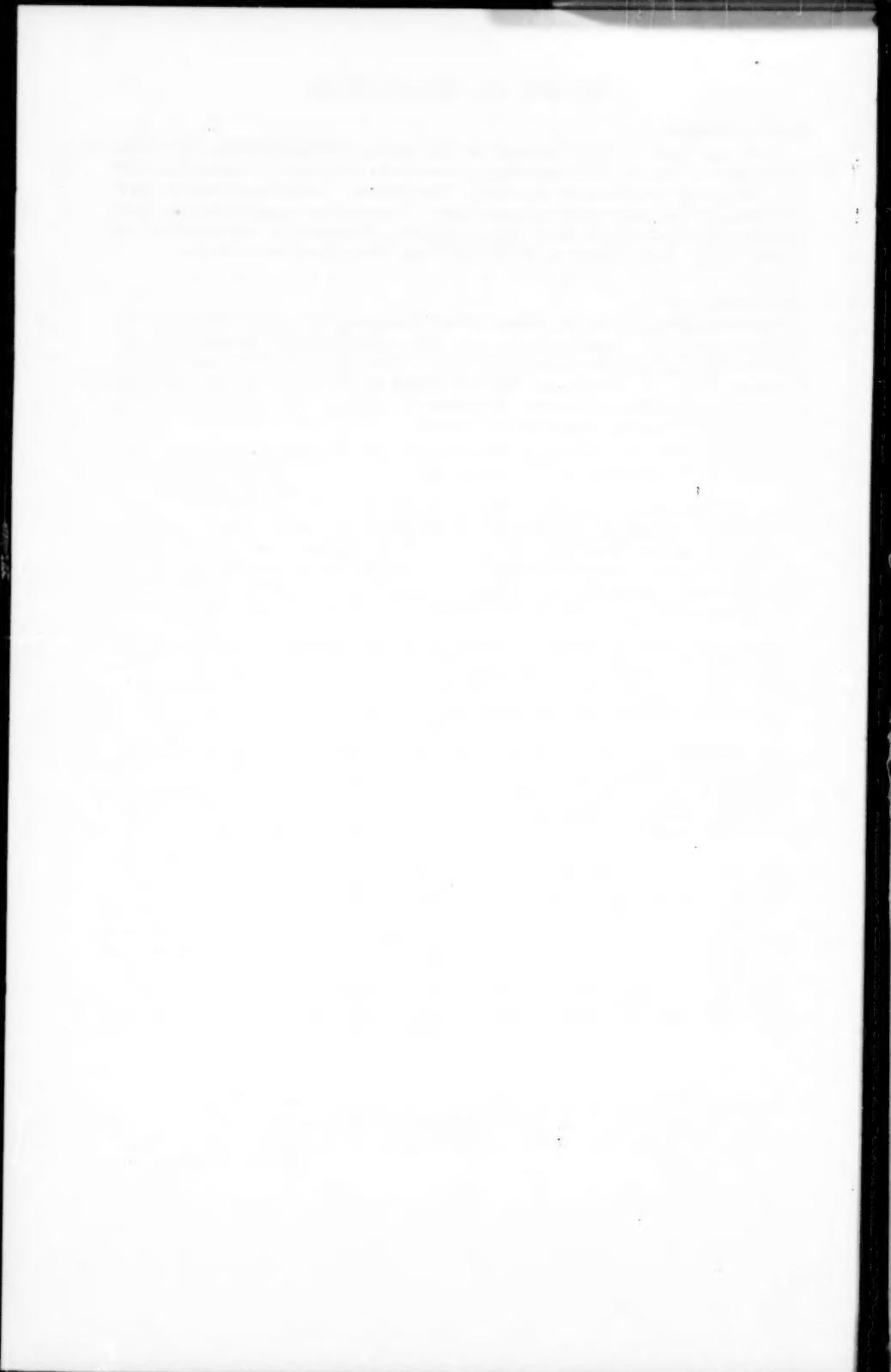
Although the Branch at Plymouth has been dissolved, a new branch has been formed at Exeter.

Problem Bureau.

There has been a slight increase in the number of inquiries for solutions. There has not been a corresponding increase in the expenses, because very few omit the stamped addressed envelope. The Bureau could handle many more problems, and is ready to tackle anything. Some of the inquiries spring from unsoundness of questions set in examinations. Requests for solutions should be sent to Dr. G. A. Carreau, 90 Wyatt Park Road, London, S.W. 2.

Officers and Council.

Council wishes to offer, on behalf of all members, its sincere thanks to the President, Mr. G. L. Parsons, and to the Officers for the work which they have done for the Association during the year. Especial tribute is expressed to Professor T. A. A. Broadbent who, as stated earlier in the report, has now retired from the office of Editor. Professor Broadbent will continue as a Vice-President and thus be a member of Council.



Euclid's Elements

EDITED BY SIR THOMAS L. HEATH

The thirteen books of Euclid's Elements translated from the text of Heiberg with an introduction and commentary by Sir Thomas L. Heath. A reprint of the second, revised edition.

Three volumes, 90s. net the set

Linear Transformations in *n*-Dimension Vector Space

H. L. HAMBURGER & M. E. GRIMSHAW

"This book gives a very readable account of the theory, particularly of the reduction to canonical forms, of linear operators in *n*-dimensional linear spaces over the complex field . . . an excellent account of its subject."—THE MATHEMATICAL GAZETTE. *A reprint is now available.*

32s. 6d. net

An Introduction to Stochastic Processes

M. S. BARTLETT

"A comprehensive summary of recent work on the applications of stochastic processes and an authoritative review of the analytical techniques required."—THE MATHEMATICAL GAZETTE.
"A characteristically original, impressive and important contribution to the subject, and deserves to be widely read."—NATURE. *A reprint is now available.*

35s. net

CAMBRIDGE UNIVERSITY PRESS

BENTLEY HOUSE, 200 EUSTON ROAD, LONDON, N.W.1

OXFORD GRADED ARITHMETIC PRACTICE

By D. A. HOLLAND

Mathematics Master at Forest Lodge Junior School, Leicester

1. Addition	5. Money: Addition and Subtraction
2. Subtraction	6. Money: Multiplication and Division
3. Short Multiplication	7. Weights and Measures (<i>In preparation</i>)
4. Short Division	8. Fractions (<i>In preparation</i>)

PUPILS' Books, Pp. 64. Manilla covers, each 2s. 6d.

TEACHER'S BOOKS, with ANSWERS in red.

Manilla covers, each 3s. 6d. net

This scheme provides practice work on the basic process and number combinations for the mental age range 7 to 11-plus; with ample material for revision.

It provides complete analysis of addition, subtraction, multiplication, and division. Each page represents one step forward towards complete mastery of the process.

It provides all the material needed for diagnosis of individual difficulties.
It shows the exact place to start remedial teaching.

It has all the material needed for oral practice and speed drills.

A full description of recommended methods is given. Every

A full description of recommended methods is given. Every page is specially arranged for the child's ease in working and the teacher's convenience in teaching and marking.

'The format of the books in this series is excellent from the teacher's point of view—diagnostic and remedial assessments, and marking, are all facilitated by the logical grading and arrangements of examples. . . . In both books [1 and 2] the number charts are useful. The number lists are a little unusual and are an important feature in stressing the comprehension of words and numbers. The books are suitable for practice throughout the Mental Age Range 7 to 11 plus, with the younger children for consolidating the learning of the basic processes and number facts, with the older children for achieving speed.'

Mathematical Gazette

'... The series will undoubtedly supply an invaluable aid for remedial teaching of retarded children.' *The Times Educational Supplement*

Applications for inspection copies should be addressed to the
OXFORD UNIVERSITY PRESS
EDUCATION DEPARTMENT OXFORD